Introduction in First-Order Combinatorics Providing a Conceptual Framework for Computation in Predicate Logic

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***Abstract*—In this work, we introduce general specifications for the concepts of finitary and infinitary first-order combinatorics as well as give preliminary definitions of semantic layers of model-theoretic properties connected with these combinatorics. We use only the simplest notions of first-order logic and algorithm theory together with elementary properties of signature reduction procedures and constructions of finitely axiomatizable theories known in common practice. The work represents an ideological basis and starting point for investigations on expressive power of first-order predicate logic.**

Keywords—first-order logic; computation; theory; computably axiomatizable theory; interpretation; signature reduction procedure; combinatorics.

I. INTRODUCTION

The principal problem concerning expressive power of first-order predicate logic was solved by W. Hanf [2][3], who proved that, for any computably axiomatizable theory T, there is a finitely axiomatizable theory F together with a computable isomorphism $\mu: \mathcal{L}(T) \to \mathcal{L}(F)$ between their Tarski–Lindenbaum algebras. Moreover, in the same work [3], Hanf gives a direct formula that presents the isomorphism type of the Tarski–Lindenbaum algebra $\mathcal{L}(PC(\sigma))$ of predicate calculus $PC(\sigma)$ of a finite rich signature σ. The works of Hanf–Myers [4] and Myers [6] introduce a method of constructing computable isomorphisms between $\mathcal{L}(PC(\sigma_1))$ and $\mathcal{L}(PC(\sigma_2))$, where σ_1 and σ_2 are arbitrary finite rich signatures.

Subsequent work of Myers [7] describes an enhanced isomorphism between the Tarski–Lindenbaum algebras $\mathcal{L}(PC(\sigma_2))$ and $\mathcal{L}(PC(\sigma_3))$, where signature σ_2 consists of a single binary predicate, while σ_3 consists of a single ternary (or n-ary, $n > 3$) predicate. He builds a computable isomorphism $\mu: \mathcal{L}(PC(\sigma_2)) \to \mathcal{L}(PC(\sigma_3))$ such that, for any complete extension T' of $PC(\sigma_2)$ and corresponding complete extension S' of $PC(\sigma_3)$, $S' = \mu(T')$, the theories T' and S' are mutually interpretable in each other via so-called tuple-quotient interpretations (using a definable set of tuples of a finite length modulo a definable equivalence relation as a domain of the interpretation). Thereby, the corresponding completions will have rather like model-theoretic properties.

Our previous work [10] represents a universal construction of finitely axiomatizable theories controlling the structure of the Tarski–Lindenbaum algebra of a theory together with a large layer of model-theoretic properties, while the works [8] and [9] describe special methods of constructing isomorphisms between the Tarski–Lindenbaum algebras of predicate calculi of different finite rich signatures. Notice that, the methods in [8] are based on the universal construction providing computable transformation of the theory, that corresponds to the term “infinitary combinatorics”. Furthermore, the methods in [9] are based on finite-to-finite signature reduction procedures providing first-order definable transformation of the theory, that corresponds to the term “finitary combinatorics”. Therefore, a natural idea arises to use the combinatory terminology for further works in this direction.

Probably, any exact definition is impossible for the concept of combinatorics as well as for its particular cases such as “finite combinatorics” or “infinite combinatorics”. However, some specifications are possible for these concepts if to restrict ourselves to the case of the language of first-order logic. The problem to define such specifications arises just in connection with the idea to define a new approach for investigations on expressive power of first-order predicate logic. Earlier, this problem was not even posed at all while the methods of first-order combinatorics were considered as obvious constructions of model theory available in the common practice; furthermore, different specialists considered different meanings of the term “first-order combinatorics” itself.

The given work introduces some general specifications to finitary and infinitary combinatorics. They are intended to be used during investigations on the problem of characterization of the Tarski–Lindenbaum algebras of predicate calculi of finite rich signatures; these algebras should be considered as generalized, i.e., enhanced with an assignment function within the finitary or infinitary semantic layer of model-theoretic properties. At such an approach, finitary first-order methods represent the finite (one can say, combinatorial) level of computation, while infinitary first-order methods represent the algorithmic level of computation in first-order predicate logic.

II. PRELIMINARIES

We consider theories in first-order predicate logic *with equality* and use general concepts of logic, model theory, algorithm theory, and constructive models found in Rautenberg [11], Hodges [5], Rogers [12], Goncharov and Ershov [1]. Generally, *incomplete* theories are considered.

A finite signature is called *rich* if it contains at least an n-ary predicate or function symbol for $n > 1$, or two unary function symbols. In this work, the signatures are considered only, which admit Gödel's numbering of the formulas. Such a signature is called *enumerable*. In writing of a signature, capital letters are used for predicates, small letters for functions and constants, and superscripts specify arities of appropriate symbols. If $\mathfrak{M}$ is a model, $|\mathfrak{M}|$ stands for the universe set of $\mathfrak{M}$. If T is a theory, by $Mod(T)$, we denote the class of all models of T. The Tarski–Lindenbaum algebra of theory T over formulas without free variables is denoted by $L(T)$, while $\mathcal{L}(T)$ stands for the Tarski–Lindenbaum algebra $L(T)$ considered together with a Gödel numbering γ such that the concept of a computable isomorphism becomes applicable to such objects. Such isomorphisms between the Tarski–Lindenbaum algebras of theories were initially considered by Hanf [2].

Let T be a theory of signature σ and $\sigma' \subseteq \sigma$. An m-ary relation P^m is called *first-order definable* in T relative to σ' if there is a formula $\varphi(x_1, \dots, x_m)$ of signature σ' such that

$$T \vdash P(x_1, \dots, x_m) \leftrightarrow \varphi(x_1, \dots, x_m).$$

Relation P is called *$\exists \cap \forall$-definable* in T relative to σ', if there are formulas $\theta(x_1, \dots, x_m)$ and $\theta'(x_1, \dots, x_m)$ of signature σ', such that $\theta(x_1, \dots, x_m)$ is an $\exists$-formula, $\theta'(x_1, \dots, x_m)$ is a $\forall$-formula, and two following conditions are satisfied:

$$T \vdash P(x_1, \dots, x_m) \leftrightarrow \theta(x_1, \dots, x_m),$$

$$T \vdash P(x_1, \dots, x_m) \leftrightarrow \theta'(x_1, \dots, x_m).$$

Particularly, the formula $(\forall x_1 \dots x_m)(\theta(x_1, \dots, x_m) \leftrightarrow \theta'(x_1, \dots, x_m))$ must be true in the theory T. Similar definitions also apply for functions and constants instead of the relation P.

Theories T and S are called *first-order equivalent* or *isomorphic*, written as $T \approx S$, if S can be obtained from T by a finite number of operations of renaming signature symbols and by adding and eliminating those signature symbols that are first-order definable in terms of other signature symbols. Theories T and S are called *first-order $\exists \cap \forall$-equivalent* or *algebraically isomorphic*, written as $T \approx_a S$, if S can be obtained from T by a finite number of operations of renaming signature symbols and by adding and eliminating those signature symbols that are $\exists \cap \forall$-definable in terms of other signature symbols. Obviously, we have $T \approx_a S \Rightarrow T \approx S$ for arbitrary theories T and S.

An arbitrary set $\mathfrak{p}$ of complete theories of enumerable signatures which is closed under $\approx$ is said to be a *model property*, while a set $\mathfrak{p}$ of complete theories closed under $\approx_a$ is said to be an *algebraic property*. Both types of properties are called *model-theoretic* properties. Examples of model-theoretic properties of model type: "theory has a prime model", "theory is not stable". An example of property of algebraic type: "theory is model complete". By AL, we denote the set of all model-theoretic properties of algebraic type, while ML stands for the set of all properties of model type; the inclusion $ML \subseteq AL$ is obvious. An arbitrary collection L of model-theoretic properties is said to be a *semantic layer*. A set $L \subseteq ML$ is called a *model semantic layer*, while a set $L \subseteq AL$ is called an *algebraic semantic layer*. Notice that, any model semantic layer can be regarded as an algebraic layer. In the case when there is a computable isomorphism $\mu: \mathcal{L}(T) \to \mathcal{L}(S)$ preserving any model-theoretic properties within a layer L, the theories T and S are said to be *semantically similar* over the layer L, symbolically written as $T \equiv_L S$.

A. Demonstration of the relation of semantic similarity

It is a simple exercise to construct a computably axiomatizable theory T satisfying the following properties: T is decidable, the set of all complete extensions of T, called its *Stone space*, consists of a countable sequence $T_k, k \in \mathbb{N} \cup \{\omega\}$, such that, each of the theories $T_0, T_1, T_2, \dots$ is a stable theory without prime models and is finitely axiomatizable over T, while T_ω is not finitely axiomatizable over T, it is not stable and has a prime model. Applying the universal construction, [10,Th.0.6.1], we can find a finitely axiomatizable theory F together with a computable isomorphism $\mu: \mathcal{L}(T) \to \mathcal{L}(F)$ preserving any property in the following immediately listed semantic layer of model-theoretic properties:

$$L = \{\text{"}\textit{theory is stable}\text{"}, \text{"}\textit{theory has a prime model}\text{"}\}.$$

Thereby, within the layer L, this theory F has exactly the same model-theoretic properties as T did. This example demonstrates concepts of a model-theoretic property, semantic layer, computable isomorphism between the Tarski–Lindenbaum algebras, as well as a possibility of applications of the universal construction.

B. Demonstration of model versus algebraic properties

Algebraic-type properties are thinner in comparison with those of model-type. Often, model-type properties are considered, while sometimes, thinner algebraic-type properties are also needed. For instance, let T be the theory of discrete linear orders considered in signature $\sigma = \{<^2, \lhd^2\}$, where

$$x \lhd y \leftrightarrow (x < y) \,\&\, (\forall z)(x \le z \le y \to (x = z \,\vee\, z = y)).$$

Since $\lhd$ is first-order definable relative to $<$, we can omit predicate $\lhd$ obtaining another theory T_0 of discrete linear orders in smaller signature $\sigma_0 = \{<^2\}$. Theories T and T_0 are isomorphic with each other; particularly, we have $T \equiv_{ML} T_0$. On the other hand, T and T_0 are not algebraically isomorphic because T is model complete; thus, all its complete extensions are model complete as well; on the contrary, there is a complete extension of T_0 which is not model complete. Thereby, $T \equiv_{AL} T_0$ does not the case.

III. CARTESIAN EXTENSIONS OF THEORIES

Let σ be a signature and

$$\xi = \langle \varphi_1^{m_1}, \varphi_2^{m_2}, \dots, \varphi_s^{m_s} \rangle \quad (1)$$

be a finite sequence of formulas of this signature, where φ_k is a formula with m_k free variables. Starting from a tuple ξ and an arbitrary model $\mathfrak{M}$ of signature σ, we will construct some new model $\mathfrak{M}_1 = \mathfrak{M}\langle \xi \rangle$ of signature

$$\sigma_1 = \sigma \cup \{U^1, U_1^1, \dots, U_s^1\} \cup \{K_1^{m_1+1}, \dots, K_s^{m_s+1}\} \quad (2)$$

as follows. As a universe for the model, we take the following set

$$|\mathfrak{M}_1| = |\mathfrak{M}| \cup A_1 \cup A_2 \cup \dots \cup A_s ,$$

where the pointed out parts are pairwise disjoint. In the part $|\mathfrak{M}|$, all symbols of signature σ are defined exactly as they were defined in $\mathfrak{M}$; in remaining, these symbols are defined trivially; U is defined by $U(x) \Leftrightarrow x \in |\mathfrak{M}|$; U_k is defined by $U_k(x) \Leftrightarrow x \in A_k$; predicate K_k represents a one-to-one correspondence between the set of tuples $\{\bar{a} | \mathfrak{M} \vDash \varphi_k(\bar{a})\}$ and the set A_k . So defined model $\mathfrak{M}\langle \xi \rangle$ is said to be *Cartesian extension of* $\mathfrak{M}$ by means of sequence (1), denoted by $\mathfrak{M}\langle \varphi_1^{m_1}, \varphi_2^{m_2}, \dots, \varphi_s^{m_s} \rangle$, or $\mathfrak{M}\langle \xi \rangle$ for short. Now, we consider a theory T of signature σ, and fix signature (2) for extensions of models. Let us define a new theory T' as follows

$$T' = Th\{\mathfrak{M}\langle \xi \rangle | \mathfrak{M} \in Mod(T)\}.$$

It is said to be *Cartesian extension* of T by means of sequence of formulas (1), denoted by $T\langle \varphi_1^{m_1}, \varphi_2^{m_2}, \dots, \varphi_s^{m_s} \rangle$, or $T\langle \xi \rangle$ for short. According to the construction, the theory $T\langle \xi \rangle$ is defined uniquely up to an algebraic isomorphism of theories; moreover, an interpretation $I_{T,\xi}$ of the source theory T in the target theory $T\langle \xi \rangle$ is naturally defined.

Now, we consider a sequence of formulas of signature σ of a more common form

$$\varkappa = \langle \varphi_1^{m_1}/\varepsilon_1, \varphi_2^{m_2}/\varepsilon_2, \dots, \varphi_s^{m_s}/\varepsilon_s \rangle, \quad (3)$$

where $\varphi_k(\bar{x})$ is a formula with m_k free variables, while $\varepsilon_k(\bar{y}, \bar{z})$ is a formula with $2m_k$ free variables. By $Equiv(\varepsilon_k, \varphi_k)$, we denote a sentence stating that ε_k is an equivalence relation on the set of tuples distinguished by the formula $\varphi_k(\bar{x})$. Let us repeat the construction given above with the only difference that $(m_k + 1)$-ary predicate K_k represents a one-to-one correspondence between the quotient set $\{\bar{a} | \mathfrak{M} \vDash \varphi_k(\bar{a})\}/\hat{\varepsilon}_k$ and the set A_k, where $\hat{\varepsilon}_k(\bar{y}, \bar{z}) = \varepsilon_k(\bar{y}, \bar{z}) \vee \neg Equiv(\varepsilon_k, \varphi_k)$. The obtained theory $T\langle \varkappa \rangle$ is said to be *Cartesian-quotient extension* of T by means of sequence of formulas $\varkappa$. Similarly to the previous case, the theory $T\langle \varkappa \rangle$ is determined uniquely up to an algebraic isomorphism of theories; moreover, there is a natural interpretation $I_{T,\varkappa}$ of the source theory T in the target theory $T\langle \varkappa \rangle$.

The introduced operations are used in further definition of first-order combinatorics.

1. Statement: *Up to an algebraic isomorphism of theories, each finite-to-finite signature reduction procedure represents a particular case of Cartesian extension of theories.*

Proof. Immediately, by Beth's Definability Theorem, [5, Th. 5.5.4]. □

IV. First-order combinatorics

First, we introduce some *common specification* in a compact form.

By *first-order combinatorics*, we mean transformation methods of countable (specifically, computably axiomatizable) theories, which can change both signature and axiomatic of the theory preserving, as much as possible, its model-theoretic properties. The emphasis is on the methods definable in first-order predicate logic, while the principal goal is the maximality of the collection of preserved model-theoretic properties. Moreover, the main objective is naturalness of the accepted specification. Significance of the complex of definitions for combinatorics is considered as higher if these definitions adequately correspond to an available approach to logic (particularly, in set theory or model theory). In the case of ambiguity in the choice of some technical details, the preference should be directed to the variants of concepts simplifying the situation or providing more perfect appearance. As an initial basis for the concept of combinatorics we take the class of signature reduction procedures, which are considered as a particular case of combinatorial methods in first-order logic. The common problem is to generalize these particular methods to maximum wide natural approach in such a way that so serious term as "combinatorics" would become acceptable here.

With this, the common specification is complete.

Now, we turn to develop the common idea in a mathematical form.

A signature reduction procedure is normally applied, when we are going to transform a given theory T having an infinite or too large finite signature to some new theory S having a small finite signature. Moreover, the target theory S must inherit from the source theory T all model-theoretic properties within a given layer $L = \{\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2, \dots\}$. Generally, specifications for the signature reduction methods are subordinated to the purpose to pass from T to S as large collection of properties as possible; thus, any exotic methods of signature reduction distorting some evident model-theoretic properties should be rejected. Ordinarily, the signature reduction procedure is determined by an interpretation I of T in S preserving the demanded properties. It is possible to establish (for instance, with the back-and-forth Ehrenfeucht method), that generally, such an interpretation I defines an isomorphism of the Tarski–Lindenbaum algebras $\mu: \mathcal{L}(T) \to \mathcal{L}(S)$ passing from T to S both structure of extensions of theory and any model-theoretic properties within the layer L from complete extensions of T to corresponding complete extensions of S. In many cases, reviewing is limited by complete theories only; in this case, the Tarski–Lindenbaum algebras $\mathcal{L}(T)$ and $\mathcal{L}(S)$ are 2-element Boolean algebras.

Notice that, in technical realization, signature reduction procedures may consist of two or more separate stages. Particularly, first, a reduction from an infinite in some finite signature could be performed, while on the second stage, the obtained finite signature is reduced to the wished small finite signature. Another remark is that the universal construction of finitely axiomatizable theories (see [10, Th.0.6.1]) is a transformation from the class of computably axiomatizable theories in the class of finitely axiomatizable theories (of finite signatures); thereby, such a transformation can be considered as an improved variant of infinite-to-finite signature reduction procedure. Moreover, for the construction, the whole transformation procedure consists of so called *main stage* (performing the actual passage from a computably axiomatizable theory to a finitely axiomatizable theory) and a few auxiliary stages performing signature reductions of certain types. Practical observation shows that, the universal construction can control the same model-theoretic properties which are under control of infinite-to-finite signature reduction procedures. This definitely shows that, both signature reduction procedures and constructions of finitely axiomatizable theories should be considered jointly as an integrated complex of transformations of theories.

V. Two types of first-order combinatorics

Combinatorics of a given type is characterized by a definite set of used methods and by collection of those model-theoretic properties which are controlled by application of these methods. Since we consider combinatorics in first-order logic, the concept of a method is understood as some manner $\mathfrak{m}$ of first-order transformation of a computably axiomatizable theory T in another such theory S producing a computable isomorphism of the Tarski–Lindenbaum algebras $\mu\colon \mathcal{L}(T) \to \mathcal{L}(S)$; moreover, as a control over a model-theoretic property $\mathfrak{p}$ we mean that the isomorphism μ passes without change this property $\mathfrak{p}$ from any complete extension of T to corresponding complete extension of S. As mentioned above, some interpretation I of T in S is meant behind the isomorphism μ. Thus, by way of constructing an input theory T, it is possible to influence on properties of the target theory S within the semantic layer L of the controlled properties.

Now, we define *finite first-order combinatorics* or shortly *finitary combinatorics* as combinatorics that is determined by finitary transformation methods between first-order theories, and *infinite computable first-order combinatorics* or shortly *infinitary combinatorics* as combinatorics that is determined by effective infinitary transformation methods between first-order theories. Finite-to-finite ($f2f$) signature reduction procedures represent transformations of theory from one finite signature in another finite signature. They are examples of finitary methods in first-order logic; at the same time, some other finitary methods in this logic exist; particularly, any Cartesian-quotient (or Cartesian) extension of a theory represents a finitary first-order method. Infinite-to-finite ($i2f$) signature reduction procedures represent transformations of theory from an infinite enumerable signature in a finite signature. They are examples of infinitary methods in first-order logic; at the same time, some other infinitary methods in this logic exist; particularly, any release of the universal construction represents infinitary first-order methods transforming computably axiomatizable theories in finitely axiomatizable theories.

There are two possibilities to compare semantic layers.

2. Rule of inverse inclusion: Any relatively smaller class of methods defines the relatively larger semantic layer, i.e., if $\mathcal{M}_1$ and $\mathcal{M}_2$ are classes of transformation methods of theories, while L_1 and L_2 are the semantic layers determined by these classes, we have $\mathcal{M}_1 \subset \mathcal{M}_2 \Rightarrow L_1 \supseteq L_2$; furthermore, the union of classes of methods $\mathcal{M}_1 \cup \mathcal{M}_2$ determines the intersection of layers $L_1 \cap L_2$; the rule is formally exact.

3. Rule of representative check: We fix a large enough list $\mathcal{R}$ of commonly known model-theoretic properties, which is agreed to be considered as representative. For two semantic layers L_1 and L_2, $L_1 \overset{\mathcal{R}}{=\!=} L_2$ means that $\mathfrak{p} \in L_1 \Leftrightarrow \mathfrak{p} \in L_2$ for all $\mathfrak{p} \in \mathcal{R}$, and $L_1 \overset{\mathcal{R}}{\leqq} L_2$ means that $\mathfrak{p} \in L_1 \Rightarrow \mathfrak{p} \in L_2$ for all $\mathfrak{p} \in \mathcal{R}$; this rule represents a practical method of comparison even in the case when no possibility exists for formally exact comparison of volumes of the semantic layers; for $\mathcal{R}$, it are possible to take the join of collections of model-theoretic properties immediately listed in [9, Lem.4.2] and [10, Th.0.6.1].

Let us formulate an important relation between finitary and infinitary methods.

4. Principle of subordination of finite to infinite: If a class of transformation methods $\mathcal{M}$ is intended for definition of some version of infinitary layer, we must include in $\mathcal{M}$ all finitary methods relevant to this class; this requirement prevents unacceptable situation when an infinitary semantic layer is defined by a class of infinitary methods where some finitary methods are missed.

There is an obvious possibility to introduce the concept of *abstract infinite first-order combinatorics* as a version of infinite combinatorics with omitted requirement of computability for the passage from T to S, and thus, for the isomorphism $\mu\colon \mathcal{L}(T) \to \mathcal{L}(S)$. Since the class of abstract infinite methods is obviously wider in comparison with the class of computable infinite methods, by the rule of inverse inclusion, the semantic layer defined by computable infinite methods extends the layer defined by abstract infinite methods. This shows minor significance of the abstract approach and establishes computable infinite first-order combinatorics as the principal player in this direction of investigations.

VI. Semantic layers defined by combinatorics

Now, we specify semantic layers, which are actual in this problematic.

We introduce the following notations:

$F2f\mathcal{L} =$ the set of all model-theoretic properties of algebraic type preserved by any $f2f$ signature reduction procedure,

$I2f\mathcal{L} =$ the set of all model-theoretic properties of algebraic type preserved by any $i2f$ signature reduction procedure $\cap\, F2f\mathcal{L}$,

$Uni\mathcal{L}$ = the set of all model-theoretic properties of model type preserved by any transformation of theories defined by the universal construction of finitely axiomatizable theories $\cap\, I2f\mathcal{L} \cap F2f\mathcal{L}$,

$ACL =$ the set of all model-theoretic properties of algebraic type preserved by any Cartesian extension of any computably axiomatizable theory,

$ADL =$ the set of all model-theoretic properties of algebraic type preserved by any Cartesian-quotient extension of any computably axiomatizable theory,

$MDL =$ the set of all model-theoretic properties of model type preserved by any Cartesian-quotient extension of any computably axiomatizable theory,

$MQL =$ the set of all model-theoretic properties of model type preserved by any quasiexact interpretation from a computably axiomatizable theory to another such theory $\cap\, MDL$,

$Fin\mathcal{L} =$ the set of all model-theoretic properties of algebraic type preserved by any finitary method between computably axiomatizable theories (an ideal concept),

$Inf\mathcal{L} =$ the set of all model-theoretic properties of model type preserved by any infinitary method between computably axiomatizable theories (an ideal concept).

For infinitary layers, intersections with finitary layers $\cap\, F2f\mathcal{L}$ and $\cap\, MDL$ are added for the sake of realization the requirement of subordination of finite methods to infinite ones, while $\cap\, I2f\mathcal{L} \cap F2f\mathcal{L}$ is added because the universal construction includes intermediate stages of types $i2f$ and $f2f$. By the rule of inverse inclusion, these intersections can be equivalently realized by adding corresponding methods in the definition. The class of quasiexact interpretations, [10, Ch.5], represents a technical framework for the universal construction, while currently, an advanced definition is available for this class (in forthcoming publication).

Semantic layer $Fin\mathcal{L}$ is said to be the *truly finitary layer*, while another layer $Inf\mathcal{L}$ is the *truly infinitary layer*. Currently, these definitions are just formal (presenting some ideal concepts), since we have not provided specifications to the set of all methods for the combinatorics. Nevertheless, one can believe that these two classes of methods must exist as mathematical objects.

The scheme in Fig. 1 shows all available model-theoretic inclusions between the layers we have defined, where the relation $L_1 \equiv L_2 \Leftrightarrow_{dfn} (L_1 \subseteq L_2 \;\&\; L_1 \overset{\mathcal{R}}{=} L_2)$ is used presenting so called 'inclusion-almost-coincidence' relation. Two upper rows in the scheme represent layers of algebraic types, while its lower part represents layers of model type.

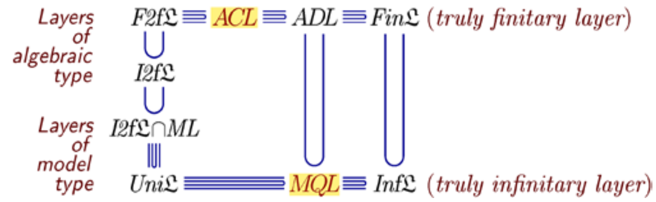

Figure 1. A dependence scheme between the semantic layers

5. Statement: *All inclusions and inclusions-almost-coincidences between the semantic layers presented in Fig. 1 take place.*

Justification. Most of the inclusions are checked immediately, using the rules of inverse inclusion and representative check. The inclusion $ACL \subseteq F2f\mathcal{L}$ is provided by Statement 1, while inclusions $MQL \subseteq ADL$ and $Inf\mathcal{L} \subset Fin\mathcal{L}$ are justified by the principle of subordination of finite layers to infinite. □

In Fig. 1, we have marked two particular semantic layers ACL and MQL. They play the role of *working* versions of the semantic layers for *finitary* and respectively *infinitary* combinatorics. It is important that the pointed out layers have complete definitions; moreover, they are most useful in applications. On the other hand, these two layers properly cover the truly finitary and respectively truly infinitary layer ensuring that practical applications with ACL and MQL are independent of investigations concerning approaches to definition of the truly semantic layers $Fin\mathcal{L}$ and $Inf\mathcal{L}$.

VII. Conclusion

Methods of finitary combinatorics represent simple and evident constructions in model theory. Methods of infinitary combinatorics are also often used. A key moment is that, each combinatorial method $\mathfrak{m}$ transforming T to S must define a computable isomorphism of the Tarski–Lindenbaum algebras $\mu\colon \mathcal{L}(T) \to \mathcal{L}(S)$. Operation of a Cartesian extension of the theory as well as other methods of finitary first-order combinatorics do not represent a great interest themselves, but they become an effective tool for investigations of the Tarski–Lindenbaum algebra of predicate calculi of finite rich signatures. However, the pointed out types of combinatorics were not provided with any strict definitions or even general agreements.

Regular references to the results known in the common practice are inappropriate within technically complicated fragments of reasoning; therefore, it is needed to introduce some formal basis for the concepts of finitary and infinitary combinatorics. This paper, providing a fundament to initial definitions concerning these combinatorics, represents a conceptual framework for the further investigations on expressive power of first-order predicate logic.

 ISBN: 978-1-61208-277-6

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