Four-State Partial Synchronizers for a Large-Scale of Processors – Symmetric Synchronizers –

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***Abstract*—The synchronization in cellular automata has been known as the Firing Squad Synchronization Problem (FSSP) since its development, where the FSSP gives a finite-state protocol for synchronizing a large scale of cellular automata. A quest for smaller state FSSP solutions has been an interesting problem for a long time. Umeo, Kamikawa and Yunès [2009] answered partially by introducing a concept of partial FSSP solutions and proposed a full list of the smallest four-state *symmetric* powers-of-2 FSSP protocols that can synchronize any one-dimensional (1D) ring cellular automata of length $n = 2^k$ for any positive integer $k \geq 1$. Afterwards, Ng [2011] also added a list of *asymmetric* FSSP partial solutions, thus completing the four-state powers-of-2 FSSP partial solutions. The number four is the lower bound in the class of FSSP protocols. A question: are there any other four-state partial solutions? remained. In this paper, we answer the question by proposing a new class of the smallest symmetric four-state FSSP protocols that can synchronize any 1D ring of length $n = 2^k - 1$ for any positive integer $k \geq 2$. We show that the class includes a rich variety of FSSP protocols that consists of 39 *symmetric* solutions, ranging from minimum-time to linear-time in synchronization steps. In addition, we make an investigation into several interesting properties of these partial solutions, such as swapping general states and a duality property between them.**

Keywords*—*cellular automata; FSSP; synchronization.

I. INTRODUCTION

We study a synchronization problem that gives a finite-state protocol for synchronizing a large scale of cellular automata. The synchronization in cellular automata has been known as the Firing Squad Synchronization Problem (FSSP) since its development, in which it was originally proposed by J. Myhill in Moore [6] to synchronize some/all parts of self-reproducing cellular automata. The FSSP has been studied extensively for more than fifty years in [1]-[12].

The minimum-time (i.e., $(2n-2)$-step) FSSP algorithm was developed first by Goto [4] for synchronizing any one-dimensional (1D) array of length $n \geq 2$. The algorithm needed many thousands of internal states for its realization. Afterwards, Waksman [11], Balzer [1], Gerken [3] and Mazoyer [5] also developed a minimum-time FSSP algorithm and reduced the number of states realizing the algorithm, each with 16, 8, 7 and 6 states. On the other hand, Balzer [1], Sanders [8] and Berthiaume et al. [2] have shown that there exists no four-state synchronization algorithm. Thus, an existence or non-existence of five-state FSSP protocol has been an open problem for a long time. Umeo, Kamikawa and Yunès [9] answered partially by introducing a concept of *partial versus full* FSSP solutions and proposing a full list of the smallest four-state symmetric powers-of-2 FSSP partial protocols that can synchronize any 1D ring cellular automata of length $n = 2^k$ for any positive integer $k \geq 1$. Afterwards, Ng [7] also added a list of asymmetric FSSP partial solutions, thus completing the four-state powers-of-2 FSSP partial solutions. A question: are there any other four-state partial solutions? remained.

In this paper, we answer the question by proposing a new class of the smallest four-state FSSP protocols that can synchronize any 1D ring of length $n = 2^k - 1$ for any positive integer $k \geq 2$. We show that the class includes a rich variety of FSSP protocols that consists of 39 symmetric solutions, ranging from minimum-time to linear-time in synchronization steps. In addition, we make an investigation into several interesting properties of these partial solutions, such as swapping general states and a duality between them.In Section 2, we give a description of the 1D FSSP on rings and review some basic results on ring FSSP algorithms. Section 3 presents a new class of the symmetric partial solutions for rings. Section 4 gives a summary and discussions of the paper.

II. FIRING SQUAD SYNCHRONIZATION PROBLEM ON RINGS

A. Definition of the FSSP on Rings

The FSSP on rings is formalized in terms of the model of cellular automata. Figure 1 shows a 1D ring cellular automaton consisting of n cells, denoted by C_i, where $1 \leq i \leq n$. All cells are identical finite state automata. The ring operates in lock-step mode such that the next state of each cell is determined by both its own present state and the present states of its right and left neighbors. All cells (*soldiers*), except one cell, are initially in the *quiescent* state at time $t = 0$ and have the property whereby the next state of a quiescent cell having quiescent neighbors is the quiescent state. At time $t = 0$ the cell C_1 (*general*) is in the *fire-when-ready* state, which is an initiation signal to the ring.

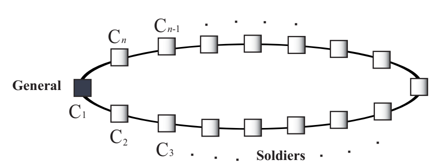


Fig. 1. One-dimensional (1D) ring cellular automaton

The FSSP is stated as follows: given a ring of n identical cellular automata, including a *general* cell which is activated at time $t = 0$, we want to give the description (state set and next-state transition function) of the automata so that, *at some future* *time*, all of the cells will *simultaneously* and, *for the first time*, enter a special *firing* state. The set of states and the next-state transition function must be independent of n. Without loss of generality, we assume $n \geq 2$. The tricky part of the problem is that the same kind of soldier having a fixed number of states must be synchronized, regardless of the length n of the ring.

A formal definition of the FSSP on ring is as follows: a cellular automaton $\mathcal{M}$ is a pair $\mathcal{M} = (\mathcal{Q}, \delta)$, where

1) $\mathcal{Q}$ is a finite set of states with three distinguished states $\mathtt{G}$, $\mathtt{Q}$, and $\mathtt{F}$. $\mathtt{G}$ is an initial general state, $\mathtt{Q}$ is a quiescent state, and $\mathtt{F}$ is a firing state, respectively.
2) δ is a next state function such that $\delta : \mathcal{Q}^3 \rightarrow \mathcal{Q}$.
3) The quiescent state $\mathtt{Q}$ must satisfy the following conditions: $\delta(\mathtt{Q}, \mathtt{Q}, \mathtt{Q}) = \mathtt{Q}$.

A ring cellular automaton $\mathcal{M}_n$ of length n, consisting of n copies of $\mathcal{M}$, is a 1D ring whose positions are numbered from 1 to n. Each $\mathcal{M}$ is referred to as a cell and denoted by C_i, where $1 \leq i \leq n$. We denote a state of C_i at time (step) t by $\mathtt{S}_i^t$, where $t \geq 0, 1 \leq i \leq n$. A *configuration* of $\mathcal{M}_n$ at time t is a function $\mathcal{C}^t : [1, n] \rightarrow \mathcal{Q}$ and denoted as $\mathtt{S}_1^t\mathtt{S}_2^t$ .... $\mathtt{S}_n^t$. A *computation* of $\mathcal{M}_n$ is a sequence of configurations of $\mathcal{M}_n$, $\mathcal{C}^0$, $\mathcal{C}^1$, $\mathcal{C}^2$, ..., $\mathcal{C}^t$, ..., where $\mathcal{C}^0$ is a given initial configuration. The configuration at time $t+1$, $\mathcal{C}^{t+1}$, is computed by synchronous applications of the next transition function δ to each cell of $\mathcal{M}_n$ in $\mathcal{C}^t$ such that:

$\mathtt{S}_1^{t+1} = \delta(\mathtt{S}_{n-1}^t, \mathtt{S}_1^t, \mathtt{S}_2^t)$, $\mathtt{S}_i^{t+1} = \delta(\mathtt{S}_{i-1}^t, \mathtt{S}_i^t, \mathtt{S}_{i+1}^t)$, for any i, $2 \leq i \leq n-1$, and $\mathtt{S}_n^{t+1} = \delta(\mathtt{S}_{n-1}^t, \mathtt{S}_n^t, \mathtt{S}_1^t)$.

A *synchronized configuration* of $\mathcal{M}_n$ at time t is a configuration $\mathcal{C}^t$, $\mathtt{S}_i^t = \mathtt{F}$, for any $1 \leq i \leq n$.

The FSSP is to obtain an $\mathcal{M}$ such that, for any $n \geq 2$,

1) A synchronized configuration at time $t = T(n)$, $\mathcal{C}^{T(n)} = \overbrace{\mathtt{F}, \cdots, \mathtt{F}}^{n}$ can be computed from an initial configuration $\mathcal{C}^0 = \mathtt{G}\overbrace{\mathtt{Q}, \cdots, \mathtt{Q}}^{n-1}$.
2) For any t, i such that $1 \leq t \leq T(n) - 1$, $1 \leq i \leq n, \mathtt{S}_i^t \neq \mathtt{F}$.

B. Full vs. Partial Solutions

One has to note that any solution in the original FSSP problem is to synchronize any array of length $n \geq 2$. We call it **full** solution. Berthiaume et al. [2] presented an eight-state full solution for the ring. On the other hand, Umeo, Kamikawa, and Yunès [9] and Ng [7] constructed a rich variety of 4-state protocols that can synchronize some infinite set of rings, but not all. We call such protocol **partial** solution. Here, we summarize recent developments on small state solutions in the ring FSSP. Berthiaume, Bittner, Perkovic, Settle, and Simon [2] gave time and state lower bounds for the ring FSSP, described in Theorems 1, 2, and 3, below.

Theorem 1 (Time Lower Bound) The minimum time in which the ring FSSP could occur is no earlier than n steps for any ring of length n.

Theorem 2 There is no 3-state **full** solution to the ring FSSP.

Theorem 3 There is no 4-state, symmetric, minimal-time **full** solution to the ring FSSP.

Umeo, Kamikawa, and Yunès [9] introduced a class of partial solutions to the FSSP and showed that there exist 17 symmetric 4-state partial solutions to the ring FSSP.

Theorem 4 There exist **17 symmetric** 4-state partial solutions to the ring FSSP for the ring of length $n = 2^k$ for any positive integer $k \geq 1$.

Ng [7] added a list of 80 asymmetric 4-state solutions, this completing the powers-of-two solutions.

Theorem 5 There exist **80 asymmetric** 4-state partial solutions to the ring FSSP for the ring of length $n = 2^k$ for any positive integer $k \geq 1$.

C. A Quest for Four-State Partial Solutions for Rings

- **Four-state ring cellular automata**
  Let $\mathcal{M}$ be a four-state ring cellular automaton $\mathcal{M} = \{\mathcal{Q}, \delta\}$, where $\mathcal{Q}$ is an internal state set $\mathcal{Q} = \{\mathtt{A}, \mathtt{F}, \mathtt{G}, \mathtt{Q}\}$ and δ is a transition function such that $\delta : \mathcal{Q}^3 \rightarrow \mathcal{Q}$. Without loss of generality, we assume that $\mathtt{Q}$ is a quiescent state with a property $\delta(\mathtt{Q}, \mathtt{Q}, \mathtt{Q}) = \mathtt{Q}$, $\mathtt{G}$ is a general state, $\mathtt{A}$ is an auxiliary state and $\mathtt{F}$ is the *firing* state, respectively. The initial configuration is $\mathtt{G}\overbrace{\mathtt{QQ}, ..., \mathtt{Q}}^{n-1}$ for $n \geq 2$. We say that an FSSP solution is *symmetric* if its transition table has a property such that $\delta(x, y, z) = \delta(z, y, x)$, for any state x, y, z in $\mathcal{Q}$. Otherwise, the FSSP solution is called *asymmetric* one.

Q (Left State \ Right State)	Q	G	A
Q	Q	•	•
G	•	•	•
A	•	•	•

G (Left State \ Right State)	Q	G	A
Q	•	•	•
G	•	•	•
A	•	•	•

A (Left State \ Right State)	Q	G	A
Q	•	•	•
G	•	•	•
A	•	•	•

Fig. 2. Four-state transition table

- **A computer investigation into four-state FSSP solutions for rings**
  Figure 2 is a four-state transition table, where a symbol • shows a possible state in $\mathcal{Q} = \{\mathtt{A}, \mathtt{F}, \mathtt{G}, \mathtt{Q}\}$. Note that we have totally 4^{26} possible transition rules. We make a computer investigation into the transition rule set that might yield possible FSSP solutions. Our strategy is based on a backtracking searching. A similar technique was employed in Ng [7]. Due to the space available, we omit the details of the backtracking searching strategy. The outline of those solutions will be described in the next section.

III. Four-State Symmetric Partial Solutions

In this section, we will establish the following theorem with a help of computer investigation.

Theorem 6 There exist **39 symmetric** 4-state partial solutions to the ring FSSP for the ring of length $n = 2^k - 1$ for any positive integer $k \geq 2$.

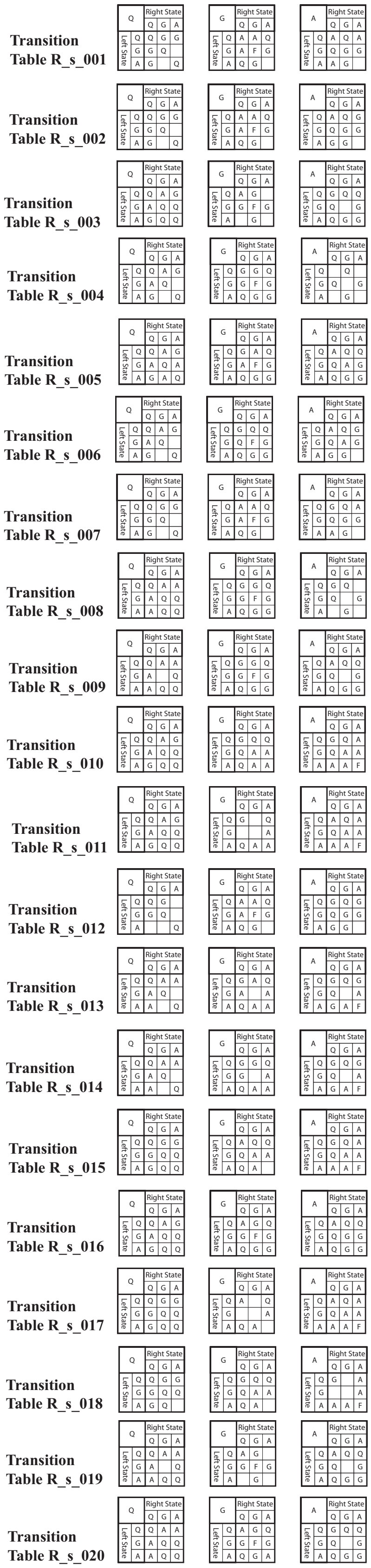


Fig. 3. Transition tables for 39 minimum-time, nearly minimum-time and non-minimum-time symmetric solutions

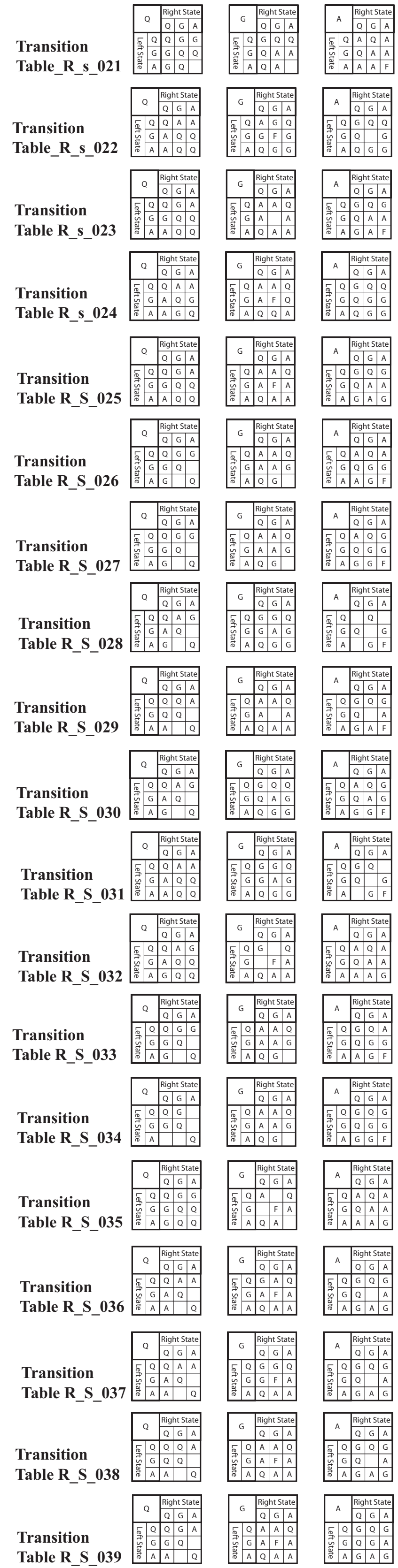


Fig. 4. Transition tables for 39 minimum-time, nearly minimum-time and non-minimum-time symmetric solutions

TABLE I. TIME COMPLEXITY AND NUMBER OF TRANSITION RULES FOR 39 SYMMETRIC PARTIAL SOLUTIONS

Symmetric Partial Solutions	Time Complexity	# of Transition Rules	Notes
R_{S_1}	$T_G(n) = T_A(n) = n$	23	
R_{S_2}	$T_G(n) = T_A(n) = n$	23	
R_{S_3}	$T_G(n) = n$	23	
R_{S_4}	$T_G(n) = n$	20	
R_{S_5}	$T_G(n) = n$	27	
R_{S_6}	$T_G(n) = n$	24	
R_{S_7}	$T_G(n) = n$	23	
R_{S_8}	$T_G(n) = T_A(n) = n$	24	
R_{S_9}	$T_G(n) = T_A(n) = n$	25	
R_{S_10}	$T_G(n) = T_A(n) = n$	27	
R_{S_11}	$T_G(n) = T_A(n) = n$	24	
R_{S_12}	$T_G(n) = n$	21	
R_{S_13}	$T_G(n) = T_A(n) = n$	23	
R_{S_14}	$T_G(n) = T_A(n) = n$	23	
R_{S_15}	$T_G(n) = T_A(n) = n$	26	
R_{S_16}	$T_G(n) = T_A(n) = n$	27	
R_{S_17}	$T_G(n) = T_A(n) = n$	23	
R_{S_18}	$T_G(n) = T_A(n) = n$	22	
R_{S_19}	$T_G(n) = T_A(n) = n$	22	
R_{S_20}	$T_G(n) = n$	26	
R_{S_21}	$T_G(n) = T_A(n) = n$	25	
R_{S_22}	$T_G(n) = T_A(n) = n$	26	
R_{S_23}	$T_G(n) = T_A(n) = n$	26	
R_{S_24}	$T_G(n) = n$	27	
R_{S_25}	$T_G(n) = T_A(n) = n + 1$	27	
R_{S_26}	$T_G(n) = T_A(n) = n + 1$	24	
R_{S_27}	$T_G(n) = T_A(n) = n + 1$	24	
R_{S_28}	$T_G(n) = n + 1$	22	
R_{S_29}	$T_G(n) = n + 1, T_A(n) = n$	23	
R_{S_30}	$T_G(n) = n + 1$	25	
R_{S_31}	$T_G(n) = T_A(n) = n + 1$	24	
R_{S_32}	$T_G(n) = T_A(n) = n + 1$	25	
R_{S_33}	$T_G(n) = n + 1$	24	
R_{S_34}	$T_G(n) = n + 1$	22	
R_{S_35}	$T_G(n) = T_A(n) = n + 1$	24	
R_{S_36}	$T_G(n) = T_A(n) = n + 1$	24	
R_{S_37}	$T_G(n) = T_A(n) = n + 1$	24	
R_{S_38}	$T_G(n) = n + 2, T_A(n) = n + 1$	24	
R_{S_39}	$T_G(n) = (3n + 1)/2, T_A(n) = n + 1$	25	

Let $R_{S\_i}, 1 \leq i \leq 39$ be a transition table for symmetric solutions obtained in this paper. We refer to the ith symmetric transition table as symmetric solution i, where $1 \leq i \leq 39$. The details are as follows:

- **Symmetric Minimum-Time Solutions:**
  We have got 24 minimum-time symmetric partial solutions operating in exactly $T(n) = n$ steps. We show their transition rules $R_{S\_i}, 1 \leq i \leq 24$ in Figures 3 and 4.
- **Symmetric Nearly Minimum-Time Solutions:**
  We have got 14 nearly minimum-time symmetric partial solutions operating in $T(n) = n + O(1)$ steps. Their transition rules $R_{S\_i}, 25 \leq i \leq 38$ are given in Figure 4. Most of the solutions, that is, solutions 25-37 operate in $T(n) = n + 1$ steps. The solution 38 operates in $T(n) = n + 2$ steps.
- **Symmetric Non-Minimum-Time Solution:**
  It is seen that one non-minimum-time symmetric partial solution 39 exists. Its time complexity is $T(n) = (3n + 1)/2$. The transition rule R_{S_39} is given in Fig. 4.

In Table I, we give the time complexity and number of transition rules for each symmetric solution.

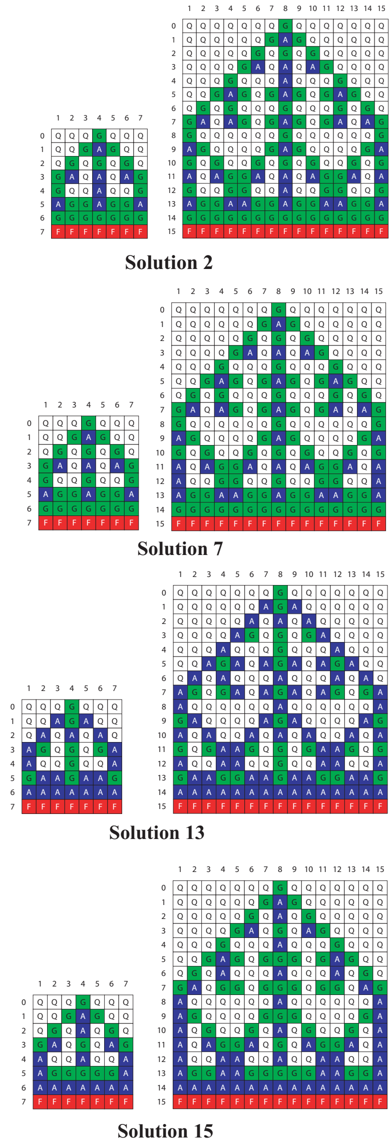


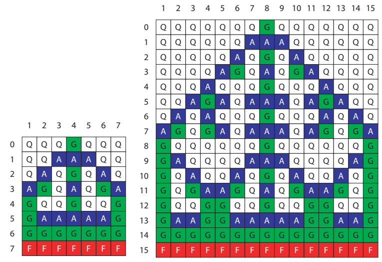
Fig. 5. Snapshots on 7 and 15 cells for symmetric solutions 2, 7, 13, and 15

Here, we give some snapshots on 7 and 15 cells for minimum-time, nearly minimum-time and non-minimum-time FSSP solutions, respectively, in Figures 5, 6, and 7.

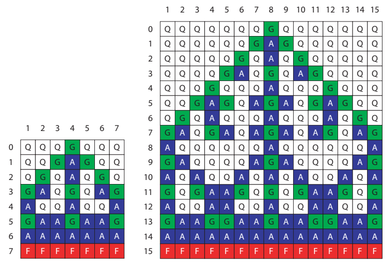
Now, we give several interesting observations obtained for the rule set.

Observation 1 (Swapping General States)
It is noted that some solutions have a property that both of the states G and A can be an initial general state without introducing any additional transition rules and yield successful synchronizations from each general state.

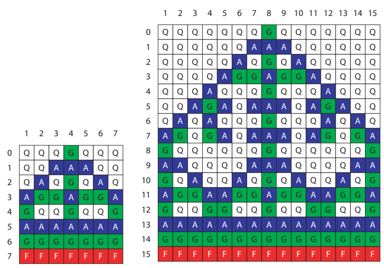
For example, solution 1 can synchronize any ring of length $n = 2^k - 1, k \geq 2$ in $T(n) = n$ steps from both an initial configuration $\mathtt{G}\overbrace{\mathtt{Q}, \cdots, \mathtt{Q}}^{n-1}$ and $\mathtt{A}\overbrace{\mathtt{Q}, \cdots, \mathtt{Q}}^{n-1}$, respectively. Let $T_{G-R_{S\_i}}(n)$ (or simply $T_G(n)$, if the rule number is specified) and $T_{A-R_{S\_i}}(n)$ ($T_A(n)$) be synchronization steps staring the solution R_{S_i} from the state G and A, respectively, for rings



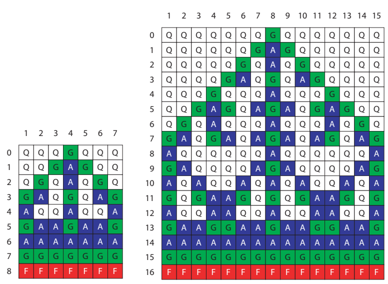
Solution 20



Solution 23



Soplution 24



Solution 25

Fig. 6. Snapshots on 7 and 15 cells for symmetric solutions 20, 23, 24, and 25

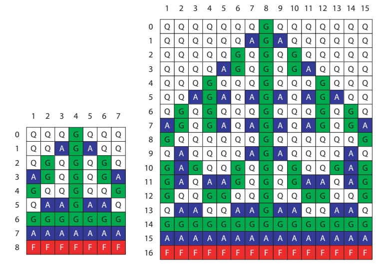
of length n. Then, we have $T_{G-R_{S\_1}}(n) = T_{A-R_{S\_1}}(n) = n$.

In Fig. 8, we show some synchronized configurations on 3, 7, and 15 cells with a general G (left) and A (right), respectively, for the solution 1. The observation doesn't always hold for all symmetric rules. For example, the solution 3 can synchronize any ring of length $n = 2^k - 1, k \geq 2$ in $T(n) = n$ steps from the general state G, but not from the state A.

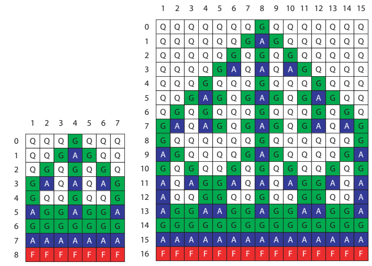
The Observation 1 yields the following *duality* relation among the four-state rule sets.

Observation 2 (Duality)
Let x and y be any four-state FSSP solution for rings and x is obtained from y by swapping the states G and A in y and vice versa. We say that the two rules x and y are dual concerning the states G and A. The relation is denoted as $x \leftrightarrows y$. We have:

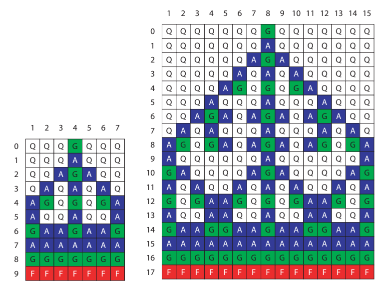
$$\begin{gathered} R_{S\_1} \leftrightarrows R_{S\_14},\ R_{S\_2} \leftrightarrows R_{S\_13}, \\ R_{S\_8} \leftrightarrows R_{S\_17},\ R_{S\_9} \leftrightarrows R_{S\_21}, \\ R_{S\_10} \leftrightarrows R_{S\_16},\ R_{S\_15} \leftrightarrows R_{S\_22}, \\ R_{S\_18} \leftrightarrows R_{S\_19},\ R_{S\_26} \leftrightarrows R_{S\_37}, \\ R_{S\_27} \leftrightarrows R_{S\_36},\ R_{S\_31} \leftrightarrows R_{S\_35}. \end{gathered}$$



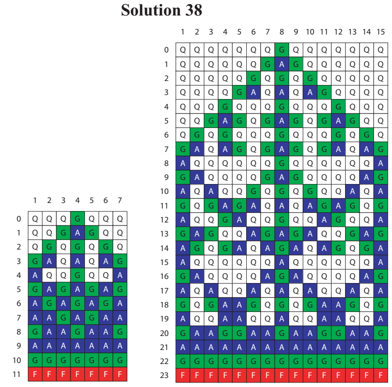
Solution 30



Solution 33



Solution 38

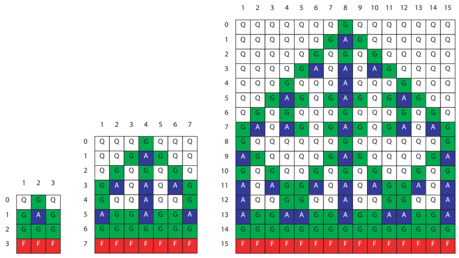


Solution 39

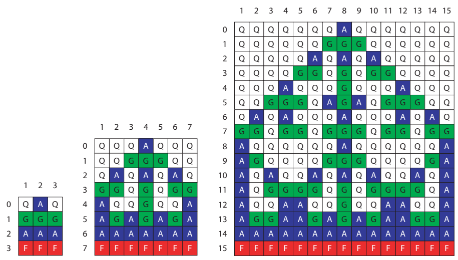
Fig. 7. Snapshots on 7 and 15 cells for symmetric solutions 30, 33, 38, and 39

IV. Summary and Discussions

A quest for smaller state FSSP solutions has been an interesting problem for a long time. We have answered to the



Solution 1 with a general-state G



Solution 1 with a general-state A

Fig. 8. Synchronized configurations on 3, 7, and 15 cells with a general-state G (upper) and A (lower), respectively, for the Solution 1

question by proposing a new class of the smallest four-state FSSP protocols that can synchronize any 1D ring of length $n = 2^k - 1$ for any positive integer $k \geq 2$. We show that the class includes a rich variety of FSSP protocols that consists of 39 symmetric solutions, ranging from minimum-time to linear-time in synchronization steps. Some interesting properties in the structure of 4-state partial solutions have been discussed. We strongly believe that no smallest solutions exist other than the ones proposed for length 2^k rings in Umeo, Kamikawa and Yunès [9] and Ng [7] and for rings of length $2^k - 1$ in this paper. A question: how many 4-state partial solutions exist for arrays (open-rings)? remains open. We think that there would be a large number of the smallest 4-state partial solutions for arrays. Its number would be larger than several thousands. The structure of the 4-state array partial synchronizers is far more complex than the 4-state ring partial synchronizers.

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