Topological Approach to Image Reconstruction in Electrical Impedance Tomography

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***Abstract*—In this paper, we investigate the application of the topological derivative in combination with the level set method for the topology optimization. The level set method and the gradient technique are based on shape and topology optimization approach to the Electrical Impedance Tomography problems with piecewise constant conductivities. The Finite Element Method and the Boundary Element Method have been used to solve the forward problem. The cost of our numerical algorithm is moderate since the shape is captured on a fixed mesh. The proposed solution algorithm is initialized by using topological sensitivity analysis. Shape derivatives and topological derivatives have been incorporated with the level set method to investigate shape optimization problems. Then it relies on the notion of shape derivatives to update the shape of the domains where conductivity takes different values. The shape derivative measures the sensitivity of boundary perturbations while the topological derivative measures the sensitivity of creating a small object in the interior domain. The coupled algorithm is a relatively new procedure to overcome this problem.**

Keywords-Image Reconstruction; Inverse Problem; Level Set Method; Optimization Methods.

I. INTRODUCTION

Numerical methods of the shape and the topology optimization were based on the level set representation and the shape differentiation [8][10]. Level set methods have been applied very successfully in many areas of the scientific modelling, for example in propagating fronts and interfaces [6][7][12][13]. Therefore, they are used to study shape optimization problems. Instead of using the physically driven velocity, the level set method typically moves the surfaces by the gradient flow of an so-called energy functional [1]. These approaches based on shape sensitivity include the elastic boundary design. There are two features that make these methods suitable for the topology optimization. The structure is represented by an implicit function such that its zero level set defines the boundary of the object. This function is often discretized on a regular grid that conveniently coincides with the finite or boundary element mesh used for structural analysis. The next valid feature is the simple update of the implicit function using the Hamilton-Jacobi equation [8], where the velocity function is determined by the shape sensitivity of the structure. These properties enable natural topology changes. The discussed technique can be applied to the solution of inverse problems in the Electrical Impedance Tomography [6][7][10][11][14].

In this work, there were implemented the novel algorithms to identify unknown conductivities. The purpose of the presented method is obtaining the better image reconstruction than gradient methods. We also want to accelerate the iterative process by using different shapes of the zero level set functions.

In the second section, we present some information about Electrical Impedance Tomography. In the third section, discussion of numerical methods is given, and in the fourth section, numerical results are shown. The last section contains conclusions.

II. ELECTRICAL IMPEDANCE TOMOGRAPHY

The Electrical Impedance Tomography (EIT) is a non-destructive imaging technique which has various applications. Its purpose is to reconstruct the conductivity of hidden objects inside a medium with the help of boundary field measurements. Efficient algorithms for solving forward and inverse problems have to be developed in order to use this approach for practical tasks. Moreover, it is necessity to improve performance of selected numerical methods. Typical problem in EIT requires the identification of the unknown internal area from near-boundary measurements of the electrical potential. It is assumed that the value of the conductivity is known in subregions whose boundaries are unknown. The forward problem in EIT is described by following partial differential equation:

$$\nabla \cdot (\gamma \nabla u) = 0, \tag{1}$$

where γ denotes conductivity. Symbol u represents electrical potential. Function u is taken under Dirichlet condition [7] in boundary points adjacent to electrodes and Neumann condition [7] on remaining part of the boundary. The problem can be reduced to determination of the minimum value of the functional:

$$I[u] = \frac{1}{2}\int_{\Omega} \gamma \left|\nabla u\right|^2 \mathrm{dxdy}. \tag{2}$$

III. NUMERICAL METHODS

Our optimization algorithm relays on several numerical methods. This section is devoted to them.

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A. Boundary Element Method

Boundary Element Method (BEM) is a well known numerical technique used to solve partial differential equations [3]. In literature, there are a lot of extensions of BEM. For example, a lot of effort has been put into combining BEM and the Finite Element Method (FEM). Another example is coupling BEM with infinite elements [4][5]. It gives us the possibility to solve equations with boundaries described by open curves. In the forward problem, we start our considerations from the following formula (proper for all boundary points) [3]:

$$\frac{1}{2}u(\vec{r}_i)+\sum_{j=1}^{N}\int_{\Gamma_j}u(\vec{r})\,q^*(\vec{r},\vec{r}_i)\,d\gamma_j=\sum_{j=1}^{N}\int_{\Gamma_j}q(\vec{r})\,u^*(\vec{r},\vec{r}_i)\,d\gamma_j. \quad (3)$$

The symbol u represents electrical potential, whereas q defines its normal derivative. The Green's function [4,14] and its normal derivative are denoted by u* and q*, respectively. In (3), we have N finite boundary elements. Next, we have introduced infinite boundary elements and the governing equation (4) has been derived. This integral equation is given by:

$$\begin{aligned}&\frac{1}{2}u(\vec{r}_i)+\sum_{j=2}^{N-1}u_j\int_{\xi=-1}^{\xi=+1}q^*(\vec{r}_j(\xi),\vec{r}_i)\,d\gamma_j+\\&+u_1\int_{\xi\to-\infty}^{\xi=+1}S_\infty(\xi)\,q^*(\vec{r}_1(\xi),\vec{r}_i)\,d\gamma_1+u_N\int_{\xi=-1}^{\xi\to+\infty}S_\infty(\xi)\,q^*(\vec{r}_N(\xi),\vec{r}_i)\,d\gamma_N\\&=\sum_{j=2}^{N-1}q_j\int_{\xi=-1}^{\xi=+1}u^*(\vec{r}_j(\xi),\vec{r}_i)\,d\gamma_j+\\&+q_1\int_{\xi\to-\infty}^{\xi=+1}S_\infty(\xi)\,u^*(\vec{r}_1(\xi),\vec{r}_i)\,d\gamma_1+q_N\int_{\xi=-1}^{\xi\to+\infty}S_\infty(\xi)\,u^*(\vec{r}_N(\xi),\vec{r}_i)\,d\gamma_N.\end{aligned} \quad (4)$$

Symbol S_∞ denotes the sum of the interpolation functions with exponential decay along infinite boundary elements. One should notice that in our model there is only one open boundary curve. However, generalizations of (4) can be easy done. In mathematical model, we assume that in N – 2 nodes the normal derivatives q equal zero. Only in two nodes we set the electrical potential.

B. Level Set Method

The level set function ϕ has the following properties:

$$\begin{aligned}&\phi(\vec{r},t)=0 \text{ for } (x,y)\in\partial\Omega(t)\equiv\Gamma(t).\\&\phi(\vec{r},t)>0 \text{ for } (x,y)\in\Omega(t),\\&\phi(\vec{r},t)<0 \text{ for } (x,y)\notin\Omega(t).\end{aligned} \quad (5)$$

The motion is seen as the convection of values (levels) from the function ϕ with the velocity field $\vec{v}$. Such process is described by the Hamilton-Jacobi equation:

$$\frac{\partial\phi}{\partial t}+\vec{v}\cdot\nabla\phi=0. \quad (6)$$

Here, $\vec{v}$ is the desired velocity on the interface, and is arbitrary elsewhere. Actually, only the normal component of $\vec{v}$ is needed ($v_n\equiv\vec{v}\cdot\vec{n}\equiv\vec{v}\cdot\nabla\phi/|\nabla\phi|$), so (6) becomes:

$$\frac{\partial\phi}{\partial t}+v_n\,|\nabla\phi|=0. \quad (7)$$

We can update the level set function ϕ by solving discretized version of the Hamilton-Jacobi equation:

$$\frac{\phi^{k+1}-\phi^k}{\Delta t}+v_n^k\,|\nabla\phi^k|=0. \quad (8)$$

Transforming above equation, we get:

$$\phi^{k+1}=\phi^k-v_n^k\,|\nabla\phi^k|\,\Delta t. \quad (9)$$

The gradient of the level set function in the k-th time step ($|\nabla\phi^k|$) has been calculated by the essentially non-oscillatory (ENO) polynomial interpolation scheme. The stability of received solution is achieved by Courant-Friedreichs-Lewy condition (CFL condition):

$$\Delta t<\frac{\min(\Delta x,\Delta y)}{\max(|\vec{v}\,|)}. \quad (10)$$

Inequality (10) is satisfied by choosing the CFL number α:

$$\Delta t\,\frac{\max(|\vec{v}\,|)}{\min(\Delta x,\Delta y)}=\alpha, \quad (11)$$

where 0 < α < 1. The optimum value equals 0.9.

The calculated velocity must be extended off the interface to the whole domain. This process is called the extension of velocity and is based on the solution of the additional partial differential equation. The reference [1] suggests:

$$\frac{\partial v_n}{\partial t}+S(\phi)\,\frac{\nabla\phi}{|\nabla\phi|}\cdot\nabla v_n=0, \quad (12)$$

where S(ϕ) is defined as following [1]:

$$S(\phi)=\frac{\phi}{\sqrt{\phi^2+\varepsilon^2}}. \quad (13)$$

In (13) |ε| << 1. Additionally, we need to extend the velocity to neighborhood of the interface, by defining velocity along normal direction (see Fig. 1).

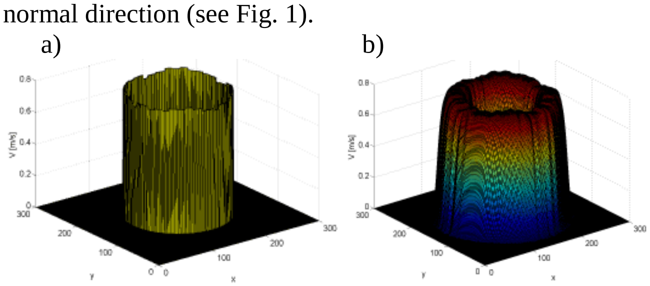

Figure 1. The velocity calculated for the first iteration step: a) - before extension; b) - after extension.

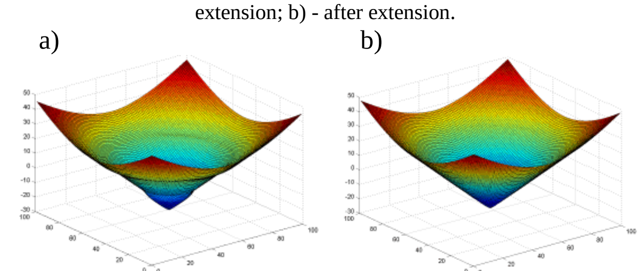

Figure 2. The level set function ϕ: a) - before reinitialization process; b) - after reinitialization process.

Reinitialization is necessary when flat or steep regions complicate the determination of the zero contour. The level set function ϕ is signed distance function if at given time for every point (x,y):

$$|\nabla \phi| = 1. \quad (14)$$

Reinitialization is based on replacing ϕ by another function that has the same zero level set, but satisfies condition (14) (see Fig. 2). This process is described by following partial differential equation [1]:

$$\frac{\partial \phi}{\partial t} + S(\phi)\,(|\nabla \phi| - 1) = 0. \quad (15)$$

Differential equation (15) is solved until a steady state is achieved. Similar to the velocity extension, a first order upwind scheme for the spatial dimension and forward Euler time discretization is used.

C. Shape derivative

The topological methods are used in order to solve the inverse problem in EIT. Very important concept for our research is so-called shape derivative. The shape derivative is often used in optimization problems [7].

Let λ be the adjoint function satisfying:

$$-\Delta\lambda = u - u_m. \quad (16)$$

The material derivative (Lagrangian derivative) $\dot{u}(x)$ is given by:

$$\dot{u}(\vec{r}) \equiv \lim_{t \to 0} \frac{u_t(\vec{r} + t\vec{v}(\vec{r})) - u(\vec{r})}{t}, \quad (17)$$

where $(x, y) \in \Omega_t$. The shape derivative is following:

$$u'(\vec{r}) \equiv \lim_{t \to 0} \frac{u_t(\vec{r}) - u(\vec{r})}{t} = \dot{u}(\vec{r}) - \vec{v}(\vec{r}) \cdot \nabla u(\vec{r}). \quad (18)$$

The steepest descent direction $\vec{v}$ is given by [7]:

$$\vec{v} = -(\nabla u \cdot \nabla \lambda)\,\vec{n}. \quad (19)$$

We need the shape derivative so that derive formula for velocity (19). The normal velocity is evaluated by using weighted least squares interpolation to get:

$$v_n^k = \nabla u^k \cdot \nabla \lambda^k + \varepsilon \kappa^k. \quad (20)$$

In next step of our procedure the level set function ϕ is updated:

$$\phi^{k+1} = \phi^k - \left(\nabla u^k \cdot \nabla \lambda^k + \varepsilon \kappa^k\right)\left|\nabla \phi^k\right| \Delta t, \quad (21)$$

where Δt is obtained from CFL condition (11).

D. Optimization algorithm

For the minimization problem iterative coupling of the level set method and the topological gradient method has been proposed. Both methods are gradient-type algorithms, and the coupled approach can be cast into the framework of alternate directions descent algorithms.

The level set method relies on the shape derivative, while the topological gradient method is based on the topological derivative. The proposed algorithm is iterative method, structured as follows:

- From the level set function at initial time, find necessary interface information.
- Use FEM or BEM to solve the equation (1) and next compute the difference of the obtained solution with the observed data.
- Solve the Poisson's equation (adjoint equation) – (16).
- Find velocity in the normal direction – (20).
- Update the level set function – (21).
- Reinitialize the level set function – (15).
- Calculate value of the objective function.

IV. Numerical Results

In the examples reported below, several numerical models with different discretization elements are presented. Additionally, we present different geometries of the conductivity distributions. We assume that the electrical conductivity of searched objects is known. The representation of the boundary shape and its evolution during an iterative reconstruction process is achieved by the level set method and the gradient method coupled together. In forward problem which is given by equation (1), we have used FEM or BEM. Additionally, different zero level set functions have been selected. Therefore, quality of the image reconstruction can be evaluated in different cases.

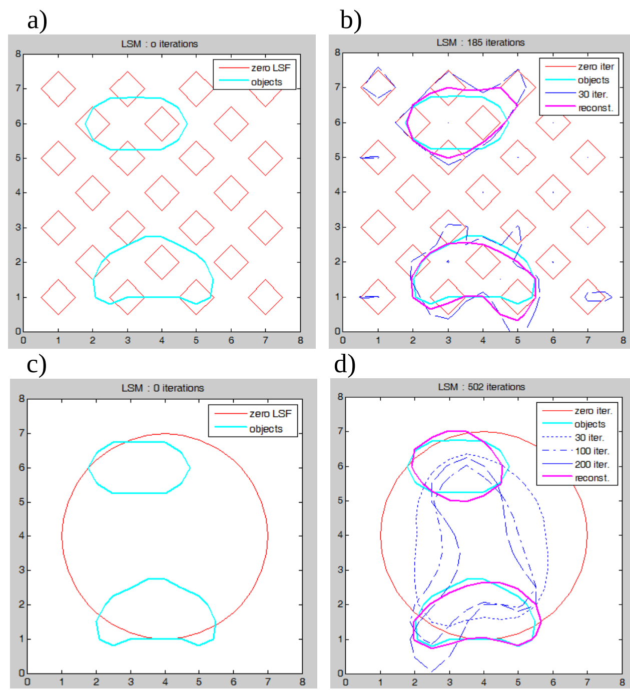

Figure 3. Images reconstruction: a), c) - the original objects and the zero contour from the level set function; b), d) - the process of the image reconstruction.

Fig. 3 shows the image reconstruction with two different groups of objects. Fig. 3a) and Fig 3c) depict two objects. The zero contour curve from the level set function is red, the following iterations are blue. The images from Fig. 3 show the original objects and reconstruction after indicated number of iterations. In the example from Fig. 3c), the zero

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level at initial step is represented by circle. The process of reconstruction is good, because the region borders are located nearly the object edges. The object function in the Fig. 3b) achieves minimum after 185 iterations, whereas the same object in the Fig. 3d) achieves the minimum after 502 iterations.

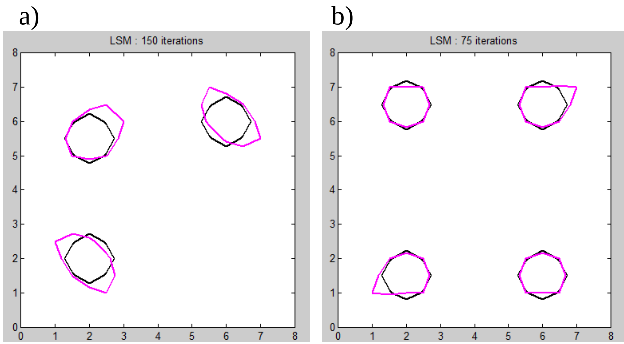

Figure 4. The image reconstruction: a) - 3 objects; b) - 4 objects.

Results of the iteration process as described above are shown in Fig. 4. Unknown structures are marked by the black line; simulated objects are marked by the pink line. For indicated numbers of iterations, the unknown structures have been found.

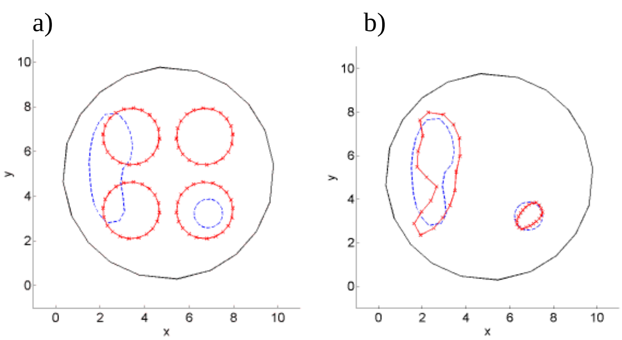

Figure 5. The black line marks the outside border of the examined structure with the internal unknown objects (marked by the blue lines). The red line marks simulated objects of zero level contours: a) - first iteration step; b) - the last iteration step (300th).

The last example of the reconstruction technique is given in Fig. 5. The image reconstructions were achieved by coupling of the level set method, the gradient technique and BEM. Our optimization algorithm works well.

V. CONCLUSIONS

An algorithm based on topological and shape derivative and the level set method have been proposed in this work. It is iterative algorithm where repeatedly the shape boundary evolves smoothly and new small objects are detected. An efficient algorithm for solving the forward and inverse problems would also improve a lot of the numerical performances of the proposed methods. In the model problem from EIT, it is required to identify unknown conductivities from near-boundary measurements of the potential. The level set function techniques have been shown to be successful to identify the unknown boundary shapes. The accuracy of the image reconstruction is better than gradient methods. The number of iterations determine the position and shape of zero level set functions. In this algorithm, we can control the process of the image reconstruction. Next advantage of this algorithm is obtaining a good quality results for the poor mesh (16x16 and 32x32 resolution). Other methods have not such properties.

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