Lyapunov's inequality for a fractional differential equation subject to a non-linear integral condition

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***Abstract*—In this paper, we investigate a Lyapunov's inequality for a non-linear fractional differential equation, subject to the non linear integral boundary condition. Such boundary conditions are different from the ones widely considered for a class of Lyapunov's inequality. The main tools are the Fubini's theorem and the construction of the Green function which corresponds to the given problem in consideration. In order to illustrate this result, we give an example where we show that the non existence of non trivial solutions of an appropriate eigenvalue fractional boundary value problem obey this necessary integral condition.**

***Index Terms*—Lyapunov's inequality, non linear boundary condition, Caputo fractional derivative, Green function.**

I. INTRODUCTION

In this paper, we present a Lyapunov's inequality for the following boundary value problem:

$$\begin{cases} ({}_cD^{\alpha}u)(t) + q(t)u(t) = 0, \\ \quad a < t < b, \quad 0 < \alpha \le 1, \\ u(a) + \mu \int_a^b u(s)q(s)\,ds = u(b), \end{cases} \tag{1}$$

where q is a continuous function defined on $[a, b]$ to R, a and b are consecutive zeros of the solution u and μ is positive. ${}_cD^{\alpha}$ stands for the Caputo derivative. It is evident to the reader to see that $u = 0$ is a trivial solution, and therefore only non-negative solutions are taken in consideration.
We prove that problem (1) has a non-trivial solution for $\alpha \in (0, 1]$ provided that the real and continuous function q satisfies

$$\int_a^b |q(t)|\,dt > \frac{\Gamma(\alpha)\mu^{\alpha}(b-a)}{\alpha(b-a+\mu)(\alpha\mu+1)^{(\alpha-1)}} \tag{2}$$

The investigation of Lyapunov's inequalities has begun very recently, where the first differential equation is based on fractional differential operators as well as that of Ferreira. In order to start with this new result, let us dwell with some references.

$$\begin{cases} u''(t) + q(t)u(t) = 0, \quad a < t < b \\ u(a) = u(b) = 0, \end{cases}$$

where a and b are consecutive zeros of u and the function $q \in C([a, b]; \mathbb{R})$. Lyapunov [1] showed the following necessary condition of existence of non-trivial solutions

$$\int_a^b |q(t)|\,dt > \frac{4}{b-a}. \tag{3}$$

Once this result is proved, similar type inequalities have been obtained for other kind of differential equations and boundary conditions see [2], [3], [4], [5].
Concerning differential equation with fractional derivative's in [6], Ferreira derived Lyapunov's inequality for the problem

$$\begin{cases} ({}_aD^{\alpha}u)(t) + q(t)u(t) = 0, \\ \quad a < t < b,\ 1 < \alpha \le 2, \\ u(a) = u(b) = 0, \end{cases} \tag{4}$$

where $q \in C([a, b], \mathbb{R})$, a and b are consecutive zeros of u, and ${}_aD^{\alpha}$ is the Riemann-Liouville fractional derivative of order $\alpha > 0$ defined for an absolute continuous function on $[a, b]$ by

$$({}_aD^{\alpha}f)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{\alpha} f(s)\,ds$$

where $n \in \mathbb{N}, n < \alpha \le n+1$ (For more details of fractional derivatives see [7]). The corresponding necessary condition of existence that he proved in [6] reads

$$\int_a^b |q(t)|\,dt > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1} = \Gamma(\alpha) \left(\frac{2^{2(\alpha-1)}}{(b-a)^{(\alpha-1)}}\right), \tag{5}$$

which in the particular case $\alpha = 2$ corresponds to Lyapunov's classical inequality (1) see [1].

Then, Ferreira [8] dealt with fractional differential boundary value problems with Caputo's derivative which is defined for a function $f \in AC^n[a, t]$ by

$$({}_a^CD^{\alpha}f)(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{\alpha} f^{(n)}(s)\,ds.$$

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For the boundary value problem

$$\begin{cases} ({}_a^C D^\alpha u)(t) + q(t)u(t) = 0, \\ \quad a < t < b,\, 1 < \alpha \leq 2, \\ u(a) = u(b) = 0, \end{cases} \tag{6}$$

where $q \in C([a,b];\mathbb{R})$ and a and b are consecutive zeros of u, Ferreira [6] proved that if (6) has a nontrivial solution, then the following necessary condition is satisfied

$$\int_a^b |q(t)|\, dt > \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{7}$$

For more details on this subject, one may see [2], [6], [8], [3], [4] and the references therein.

II. Main results

The aim of this section is to investigate the necessary condition of existence of non trivial solutions of the given boundary value problem (1). To do this, we need to use the two following auxiliary lemmas.

Lemma 1: [7], [5] Let $\alpha > 0$, then the differential equation

$${}_c D^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1}$, $c_i \in R, i = 0,1,2,...,n-1$, $n = [\alpha]+1$.

Lemma 2: [7], [5] Let $\alpha > 0$, then

$$I_c^\alpha D^\alpha h(t) = I^\alpha h(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1}$$

for some $c_i \in R, i = 0,1,2,...,n-1$, $n = [\alpha]+1$.

III. A Lyapunov-type inequality for problem (1)

In order to prove the corresponding Lyapunov-type inequality for the non linear integral fractional boundary value problem (1), let us re-write the considered problem in its equivalent integral form. Indeed, the next lemma formulates this fact.

Lemma 3: The solution u of (1) can be written in the integral form as

$$u(t) = \int_a^t G(t,s)q(s)u(s)\, ds + \int_t^b G(t,s)q(s)u(s)\, ds,$$

where the Green function $G(x,t)$ is defined by

$$\Gamma(\alpha)G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(b-s)^\alpha}{b-a\alpha\Gamma(\alpha)} \\ + \frac{(b-s)^{\alpha-1}}{\mu(b-a)\Gamma(\alpha)}, \\ a \leq s \leq t, \\ \\ -\frac{(b-s)^\alpha}{(b-a)\alpha\Gamma(\alpha)} + \frac{(b-s)^{\alpha-1}}{\mu(b-a)\Gamma(\alpha)}, \\ t \leq s \leq b. \end{cases} \tag{8}$$

$$= \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - g(t,s), & a \leq s \leq t \leq b, \\ -g(t,s) \end{cases} \tag{9}$$

where $g(t,s) := \frac{(b-s)^\alpha}{(b-a)\alpha\Gamma(\alpha)} - \frac{(b-s)^{\alpha-1}}{\mu(b-a)\Gamma(\alpha)}$, for $a \leq t \leq s \leq b$.

For the proof of Lemma 3, we use Lemma 1 and 2 to express $u(t)$ as

$$u(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + ... + c_{n-1}(t-a)^{n-1},$$

for some $c_i \in R, i = 0,1,2,...,n-1$, from one side. From another side, we have

$$I_c^\alpha D^\alpha u(t) = I^\alpha u(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1}.$$

In particular for $0 < \alpha \leq 1$, we obtain

$$u(t) = I^\alpha u(t) - c_0 = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \,-\, c_0, \tag{10}$$

where $h(s) = u(s)q(s)$, for some constant c_0 in R.
We make integration of each side of (10) between a and b, and using Fubini's integral theorem, we get

$$\int_a^b h(s)\, ds = \int_a^b \frac{(b-\tau)^\alpha}{\alpha\Gamma(\alpha)} h(\tau)d\tau \,-\, c_0(b-a). \tag{11}$$

Employing the boundary condition (1), we obtain

$$u(a) = -c_0, \tag{12}$$

and

$$u(b) = \int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \,-\, c_0. \tag{13}$$

Now combining (1), (11) and (13) we get

$$\begin{aligned} c_0 &= \frac{1}{b-a}\int_a^b \frac{(b-s)^\alpha}{\alpha\Gamma(\alpha)} h(s)ds \quad (14) \\ &- \frac{1}{\mu(b-a)}\int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds. \end{aligned}$$

Making insertion of (14) into (10) we find

$$\begin{aligned} u(t) &= \frac{1}{b-a}\int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \quad (15) \\ &- \frac{1}{b-a}\int_a^b \frac{(b-s)^\alpha}{\alpha\Gamma(\alpha)} h(s) \\ &+ \int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds. \end{aligned}$$

Therefore the Green function of the given fractional boundary problem is obtained as described in (9). The next theorem is an essential tool to prove the main theorem of the paper

Theorem 1: The Green function G satisfies:

(1) $G(t,s) \geq 0$ for all $a \leq t,\ s \leq b$.
(2) $\max_{t\,\in\,[a,b]} G(t,s) = G(b,s), \quad s \in [a,b]$,
(3) $G(b,s)$ has a unique maximum given by:

$$\max_{s\,\in\,[a,b]} G(b,s) = \frac{\alpha(b-a+\mu)(\alpha\mu+1)^{(\alpha-1)}}{\mu^\alpha(b-a)\alpha\Gamma(\alpha)},$$

provided that

$$0 < \mu(b-a) < \alpha.$$

For the proof of Theorem 1, we start by deriving the function $G(t,s)$ with respect to t, for $s \leq t$, as follows

$$\frac{\partial G}{\partial t} = -\frac{(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}, \tag{16}$$

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which is positive since $0 < \alpha \leq 1$. Thus, the Green function G is increasing as a function of t. We have

$$G(s,s) < G(t,s) < G(b,s).$$

Let us start with the right hand side of this last inequality which is $G(b,s)$, and denote it by $H(s)$. Then, we have

$$\begin{aligned} H(s) \, := G(b,s) &= \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(b-s)^{\alpha}}{(b-a)\alpha\Gamma(\alpha)} \\ &+ \frac{(b-s)^{\alpha-1}}{\mu(b-a)\alpha\Gamma(\alpha)}\Gamma(\alpha). \end{aligned}$$

Deriving H with respect to s, we get

$$H'(s) = (\alpha\mu + 1)(b-s)^{\alpha-1} - \mu(b-s)^{\alpha}.$$

We may observe that H' is equal to zero for $s = b - \frac{\alpha\mu+1}{\mu} := s^*$, and H' is positive for $s < s^*$ and negative for $s > s^*$. We conclude that the maximum of H is achieved at $s = s^*$. To this end the maximum of the Green function G is attained at $s = s^*$ and therefore

$$\max_{s \leq t} G(b,s) = G(b,s^*) = \frac{\alpha(b-a+\mu)(\alpha\mu+1)^{(\alpha-1)}}{\mu^{\alpha}(b-a)\alpha\Gamma(\alpha)}.$$

For the positivity of G, we consider $G(s,s)$ as a function of s defined by:

$$K(s) \, := G(s,s) = -\frac{(b-s)^{\alpha}}{(b-a)\alpha\Gamma(\alpha)} + \frac{(b-s)^{\alpha-1}}{\mu(b-a)\Gamma(\alpha)}.$$

Deriving K with respect to s, we obtain

$$K'(s) = \frac{\mu\alpha(b-s)^{\alpha-1} - \alpha(\alpha-1)(b-s)^{\alpha-2}}{\mu(b-a)\alpha\Gamma(\alpha)}$$

which is positive in view of $0 < \alpha \leq 1$, and $\mu > 0$. In other terms, the function K is an increasing function of s and therefore K satisfies

$$K(s) > K(a) := \frac{\mu\alpha(b-a)^{\alpha-1} - \alpha(\alpha-1)(b-a)^{\alpha-2}}{\mu(b-a)\alpha\Gamma(\alpha)}.$$

Now, in light of the assumption of the Theorem 1, $h(a)$ is a positive function in a which leads the Green function G to be positive. We note, in turn, that we derive $G(t,s)$ with respect to t for $s \geq t$, we obtain:

$$\frac{\partial G}{\partial t} = -\frac{\partial g}{\partial t} = 0. \tag{17}$$

Thus, the maximum of G is achieved at $s = s^*$ for all s, t in $[a,b]$.

Finally, we are now ready to prove the aim of this paper which is Lyapunov's inequality for the non integral boundary fractional boundary problem (1).

Theorem 2: Let u be a non trivial solution satisfying the following boundary value problem

$$\begin{cases} ({}_aD^{\alpha}u)(t) + q(t)u(t) = 0, \\ \quad a < t < b, \, 0 < \alpha \leq 1, \\ u(a) + \mu \int_a^b h(s)\, ds = u(b), \end{cases} \tag{18}$$

where a and b are two consecutive zeros of u and μ is a positive constant in R. Then (18) has a non-trivial solution provided that the real and continuous function q satisfies the following integral condition

$$\int_a^b |q(t)|\, dt > \frac{\Gamma(\alpha)\mu^{\alpha}(b-a)}{\alpha(b-a+\mu)(\alpha\mu+1)^{(\alpha-1)}}, \tag{19}$$

provided that

$$0 < \mu(b-a) < \alpha.$$

For the proof of Theorem 2, we equip the Banach space $C[a,b]$ with the Chebychev norm $||u|| = \max_{t\in[a,b]} |u(t)|$.

As

$$u(t) \, := \int_a^b G(t,s)q(s)u(s)\, ds,$$

we have

$$||u|| \, \leq \, \int_a^b \max_{t,s \,\in [a,b]} |G(t,s)|\, |q(s)|\, ds \; ||u||.$$

Then since u is a non trivial solution, in view of Theorem 1, we get

$$1 \, \leq \, \int_a^b \frac{1}{\Gamma(\alpha)} \left(\frac{\alpha(b-a+\mu)(\alpha\mu+1)^{(\alpha-1)}}{\mu^{\alpha}(b-a)} \right) |q(s)|\, ds.$$

Using the properties of G, the inequality (19) is obtained.

IV. Application

In order to illustrate Theorem 2, we give an application of Lyapunov-type inequality (19) for the following eigenvalue problem and we will get a bound for λ for which the boundary value problem in consideration has a non trivial solution. Precisely, we will show how the necessary condition of existence can be employed to determine intervals for the real zeros of the Mittag-Leffeler function.

$$\begin{aligned} E_{\alpha}(z) &= \Sigma_{k=0}^{\infty} \frac{z^k}{k\alpha + \alpha}, \\ & z \in \mathcal{C}, \quad \text{and } \mathcal{R}(\alpha) \text{ is positive}. \end{aligned} \tag{20}$$

By setting $a = \frac{1}{2}$, $b = 1$, $\mu = \alpha$, and taking into consideration the Sturm-Liouville eigenvalue problem

$$\begin{cases} ({}_{\frac{1}{2}}D^{\alpha}u)(t) + \lambda u(t) = 0, \\ \quad 0 < t < \frac{1}{2}, \quad 0 < \alpha \leq 1, \\ u(\frac{1}{2}) + \alpha \int_{\frac{1}{2}}^{1} u(s)\, ds = u(1). \end{cases} \tag{21}$$

Theorem 3: If λ is an eigenvalue of the fractional boundary value problem (21) then the following inequality holds

$$|\lambda| \, \geq \, \frac{\Gamma(\alpha)\alpha^{\alpha-1}}{(\alpha+2)(\alpha^2+1)}. \tag{22}$$

For the proof of Theorem 3, it is sufficient to use the integral inequality (19). We assume that, if λ is an eigenvalue of the boundary value problem (21), then there exists only one non-trivial solution depending on λ such that the following inequality is satisfied

$$\int_{\frac{1}{2}}^{1} |\lambda|\, dt \geq \frac{\Gamma(\alpha)\alpha^{\alpha}(b-a)}{\alpha(b-a+\alpha)(\alpha^2+1)^{\alpha}}, \tag{23}$$

or equivalently, we have

$$|\lambda| \geq \frac{\Gamma(\alpha)\alpha^{\alpha-1}}{(\alpha+2)(\alpha^2+1)}. \tag{24}$$

which completes the proof.

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