Fuzzy Weights Representation of AHP for Inner Dependence among Alternatives

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***Abstract* - The Analytic Hierarchy Process (AHP) proposed by T. L. Saaty has been widely used in decision making. Inner dependence method AHP is used for cases in which criteria or alternatives are not independent enough and using the original AHP or inner dependence AHP may cause results to lose reliability because the comparison matrix is not necessarily sufficiently consistent. In such cases, fuzzy representation for weighting criteria or alternatives using results from sensitivity analysis is useful. We present local weights of normal AHP alternatives via fuzzy sets, and then calculate modified fuzzy weights. We also get overall weights of alternatives based on certain assumptions. Results show the fuzziness of inner dependence AHP if the comparison matrix is not sufficiently consistent and individual alternatives do not have enough independence.**

Keywords - decision making; AHP; fuzzy sets; sensitivity analysis.

I. Introduction

The Analytic Hierarchy Process (AHP) proposed by T.L. Saaty in 1977 [1][2][3] is widely used in decision making, because it reflects humans feelings “naturally”. A normal AHP assumes independence among criteria and alternatives, although it is difficult to choose enough independent elements. Inner dependence method AHP [4] is used to solve this problem even for criteria or alternatives having dependence.

A comparison matrix may not, however, have enough consistency when AHP or inner dependence is used because, for instance, a problem may contain too many criteria or alternatives for decision making, meaning that answers from decision-makers, i.e., comparison matrix components, are ambiguous or fuzzy [5]. To solve this problem, we consider that weights should also have ambiguity or fuzziness. Therefore, it is necessary to represent these weights using fuzzy sets.

Our research applies sensitivity analysis [6] to inner dependence AHP to analyze how much the components of a pairwise comparison matrix influence the weights and consistency of a matrix [7]. This may enable us to show the magnitude of fuzziness in weights. We previously proposed new representation for criteria and alternatives weights in AHP [8][9], also representation for criteria weights for inner dependence, as L-R fuzzy numbers [10]. In this paper, we propose a fuzzy representation of overall alternative weights for double inner dependence structure AHP, using results from sensitivity analysis and fuzzy operations. We then represent fuzziness as a result of double inner dependence AHP when a comparison matrix among alternatives does not have enough consistency.

In Section 2, we introduce AHP and its inner dependence method. The sensitivity analyses for AHP are described in Section 3. Then the fuzzy weight representation is defined in Section 4, and Section 5 is a conclusion.

II. Inner dependence AHP

In this section, we introduce steps of normal AHP and its inner dependence method.

A. Process of Normal AHP

(Process 1) Representation of structure by a hierarchy. The problem under consideration can be represented in a hierarchical structure. The highest level of the hierarchy consists of a unique element that is the overall objective. At the lower levels, there are multiple activities (i.e., elements within a single level) with relationships among elements of the adjacent higher level to be considered. The activities are evaluated using subjective judgments of a decision maker. Elements that lie at the upper level are called parent elements while those that lie at lower level are called child elements. Alternative elements are put at the lowest level of the hierarchy

(Process 2) Paired comparison between elements at each level. A pairwise comparison matrix A is created from a decision maker's answers. Let n be the number of elements at a certain level. The upper triangular components of the comparison matrix a_{ij} ($i< j = 1,\dots,n$) are 9, 8, .. , 2, 1, 1/2, …, or 1/9. These denote intensities of importance from activity i to j. The lower triangular components a_{ji} are described with reciprocal numbers as follows

$$a_{ji} = 1/a_{ij} \tag{1}$$

in addition, for diagonal elements, let $a_{ii} = 1$. The lower triangular components and diagonal elements are occasionally omitted from the written equation as they are evident if upper triangular components are shown. The decision maker should make $n(n-1)/2$ paired comparisons at a level with n elements.

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(Process 3) Calculations of weight at each level. The weights of the elements, which represent grade of importance among each element, are calculated from the pairwise comparison matrix. The eigenvector that corresponds to a positive eigenvalue of the matrix is used in calculations throughout in this paper.

(Process 4) Priority of an alternative by a composition of weights. The composite weight can be calculated from the weights of one level lower. With repetition, the weights of the alternative, which are the priorities of the alternatives with respect to the overall objective, are finally found**.**

B. Consistency

Since components of the comparison matrix are obtained by comparisons between two elements, coherent consistency is not guaranteed. In AHP, the consistency of the comparison matrix A is measured by the following consistency index (C.I.)

$$\text{C.I.} = \frac{\lambda_A - n}{n-1}, \tag{2}$$

where n is the order of matrix A, and λ_A is its maximum eigenvalue.

It should be noted that $\text{C.I.} \geq 0$ holds. Also, if the value of C.I. becomes smaller, then the degree of consistency becomes higher, and vice versa. The comparison matrix is consistent if the following inequality holds.

$$\text{C.I.} \leq 0.1 \tag{3}$$

Also consistency ratio (C.R.) is defined as

$$\text{C.R.} = \frac{\text{C.I.}}{M}, \tag{4}$$

where M is random consistency value. However we only employ C.I., since we mainly use 4 or 5-dimensional data whose random consistency value is not far from 1

C. Inner Dependence Structure

The normal AHP ordinarily assumes independence among criteria and alternatives, although it is difficult to choose enough independent elements. Inner dependence AHP [4] is used to solve this type of problem even for criteria or alternatives having dependence.

In the method, using a dependency matrix $F=\{f_{ij}\}$, we can calculate modified weights $\boldsymbol{w}^{(n)}$ as follows,

$$\boldsymbol{w}^{(n)}=F\boldsymbol{w} \tag{5}$$

where $\boldsymbol{w}$ is weights from independent criteria or alternatives, i.e., normal weights of normal AHP and dependency matrix F is consist of eigenvectors of influence matrices showing dependency among criteria or alternatives.

If there is dependence among alternatives, we can calculate modified weights of alternatives $\boldsymbol{u}_i^{(n)}$ with only respect to criterion i. Then we composite these 2 weights to calculate overall weights of alternative k, $v_k^{(n)}$ as follow:

$$v_k^{(n)} = \sum_i^m w_i u_{ik}^{(n)}$$

where m is number of criteria.

III. Sensitivity Analyses

When we actually use AHP, it often occurs that a comparison matrix is not consistent or that there is not great difference among the overall weights of the alternatives. In these cases, it is very important to investigate how components of the pairwise comparison matrix influence on its consistency or on the weights. To analyse how results are influenced when a certain variable has changed, we can use sensitivity analysis.

In this study, we use a method that some of the present authors have proposed before. It evaluates a fluctuation of the consistency index and the weights when the comparison matrix is perturbed. It is useful because it does not change a structure of the data.

Since the pairwise comparison matrix is a positive square matrix, Perron-Frobenius theorem holds [11]. From Perron-Frobenius theorem, following theorem about a perturbed comparison matrix holds.

Theorem 1 *Let* $A = (a_{ij})$, $(i, j = 1,\ldots, n)$ *denote a comparison matrix and let* $A(\varepsilon) = A+ \varepsilon D_A$, $D_A=(a_{ij}d_{ij})$ *denote a matrix that has been perturbed. Let* λ_A *be the Frobenius root of A,* $\boldsymbol{w}$ *be the eigenvector corresponding to* λ_A, *and* $\boldsymbol{v}$ *be the eigenvector corresponding to the Frobenius root of* A'. *Then, a Frobenius root* $\lambda(\varepsilon)$ *of* $A(\varepsilon)$ *and a corresponding eigenvector* $\boldsymbol{w}(\varepsilon)$ *can be expressed as follows*

$$\lambda(\varepsilon) = \lambda_A + \varepsilon\lambda^{(1)} + o(\varepsilon), \tag{6}$$

$$\boldsymbol{w}(\varepsilon) = \boldsymbol{w} + \varepsilon\boldsymbol{w}^{(1)} + \boldsymbol{o}(\varepsilon), \tag{7}$$

where

$$\lambda^{(1)} = \frac{\boldsymbol{v}' D_A \boldsymbol{w}}{\boldsymbol{v}' \boldsymbol{w}}, \tag{8}$$

$\boldsymbol{w}^{(1)}$ *is an n-dimension vector that satisfies*

$$(A-\lambda_A I)\boldsymbol{w}^{(1)} = -(D_A - \lambda^{(1)} I)\boldsymbol{w}, \tag{9}$$

where $\boldsymbol{o}(\varepsilon)$ *denotes an n-dimension vector in which all components are* $o(\varepsilon)$.

A. Analysis for consistency of pairwise comparison

About a fluctuation of the consistency index, following corollary can be obtained from Theorem 1.

Corollary 1 *Using appropriate* g_{ij}*, we can represent the consistency index* C.I.(ε) *of the perturbed comparison matrix* $A(\varepsilon)$ *as follows*

$$\text{C.I.}(\varepsilon) = \text{C.I.} + \varepsilon \sum_{i}^{n} \sum_{j}^{n} g_{ij} d_{ij} + o(\varepsilon). \tag{10}$$

To see g_{ij} in the equation (10) in Corollary 1, how the components of a comparison matrix impart influence on its consistency can be found.

B. Analysis for weights of AHP

About the fluctuation of the weights, following corollary also can be obtained from Theorem 1.

Corollary 2 *Using appropriate* $h_{ij}^{(k)}$*, we can represent the fluctuation* $\boldsymbol{w}^{(1)}=(w_k^{(1)})$ *of the weight (i.e., the eigenvector corresponding to the Frobenius root) as follows*

$$w_k^{(1)} = \sum_{i}^{n} \sum_{j}^{n} h_{ij}^{(k)} d_{ij}. \tag{11}$$

From the equation (7) in Theorem 1, the component that has a great influence on weight $\boldsymbol{w}(\varepsilon)$ is the component which has the greatest influence on $\boldsymbol{w}^{(1)}$. Accordingly, from Corollary 2, how components of a comparison matrix impart influence on the weights, can be found, to see $h_{ij}^{(k)}$ in the equation (11).

Calculations or proofs of these theorem and corollaries are shown in [7].

IV. Fuzzy Weights Representations

A comparison matrix often has poor consistency (i.e., 0.1<C.I.<0.2) because it encompasses several criteria or alternatives. In these cases, comparison matrix components are considered to be fuzzy because they are results from human fuzzy judgment. Weights should therefore be treated as fuzzy numbers.

A. L-R Fuzzy Numbers

To represent fuzziness of weights, an L-R fuzzy number is used.

L-R fuzzy number

$$M = (m, \alpha, \beta)_{LR} \tag{12}$$

is defined as fuzzy sets whose membership function is as follows.

$$\mu_M(x) = \begin{cases} R\left(\dfrac{x-m}{\beta}\right) & (x > m), \\ L\left(\dfrac{m-x}{\alpha}\right) & (x \le m). \end{cases}$$

where $L(x)$ and $R(x)$ are shape function .

B. Fuzzy Weights of Criteria or Alternatives of normal AHP

From the fluctuation of the consistency index, the multiple coefficient $g_{ij}h_{ij}^{(k)}$ in Corollary 1 and 2 is considered as the influence on a_{ij} .

Since g_{ij} is always positive, if the coefficient $h_{ij}^{(k)}$ is positive, the real weight of criterion or alternative k is considered to be larger than w_k. Conversely, if $h_{ij}^{(k)}$ is negative, the real weight of criterion or alternative k is considered to be smaller. Therefore, the sign of $h_{ij}^{(k)}$ represents the direction of the fuzzy number spread. The absolute value $g_{ij}\ |h_{ij}^{(k)}|$ represents the size of the influence.

On the other hand, if C.I. becomes bigger, then the judgment becomes fuzzier.

Consequently, multiple C.I. $g_{ij}|h_{ij}^{(k)}|$ can be regarded as a spread of a fuzzy weight concerned with a_{ij}.

Definition 1 (fuzzy weight) Let $w_k^{(n)}$ be a crisp weight of criterion or alternative k of inner dependence model, and $g_{ij}\ |h_{ij}^{(k)}|$ denote the coefficients found in Corollary 1 and 2. If 0.1<C.I.<0.2, then a fuzzy weight $\tilde{w}_k$ is defined by

$$\tilde{w}_k = (w_k, \alpha_k, \beta_k)_{LR} \tag{13}$$

where

$$\alpha_k = \text{C.I.} \sum_{i}^{n} \sum_{j}^{n} s(-, h_{kij}) g_{ij}\ |h_{kij}|, \tag{14}$$

$$\beta_k = \text{C.I.} \sum_{i}^{n} \sum_{j}^{n} s(+, h_{kij}) g_{ij}\ |h_{kij}|, \tag{15}$$

$$s(+,h) = \begin{cases} 1, (h \ge 0) \\ 0. (h < 0) \end{cases}, \quad s(-,h) = \begin{cases} 1, (h < 0) \\ 0. (h \ge 0) \end{cases}$$

C. Fuzzy Weights for Inner dependence among Alternatives

For inner dependence structure among alternatives, we can define and calculate modified fuzzy local weights of alternatives $\tilde{\boldsymbol{u}}_i^{(n)} = (\tilde{u}_{ik}^{(n)}),\ \ k = 1, \ldots, m$ with only respect to criterion i using an dependence matrix F_A as follows,

$$\tilde{u}_{k}^{(n)} = (u_{ik}^{(n)}, \alpha_{ik}^{(n)}, \beta_{ik}^{(n)})_{LR} \tag{16}$$

where

$$\boldsymbol{u}_i^{(n)} = (u_{ik}^{(n)}) = F_A \boldsymbol{u}_i \tag{17}$$

$\boldsymbol{u}_i$ is crisp local alternative weights with only respect to criterion i and α_{ik}, β_{ik} are calculated by fuzzy multiple operations, equation(5) and definition 1.

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Fuzzy overall weights of alternative k for inner dependence among alternatives can be also calculated as follows, using fuzzy multiple $\otimes$ and fuzzy summation operations:

$$\tilde{v}_k^{(n)} = \sum_i^m w_i \otimes \tilde{u}_{ik}^{(n)}$$

where $\boldsymbol{w} = (w_i)$ is crisp weights of criteria.

Then we can evaluate fuzzy overall weights of alternatives with their centers and spreads.

V. CONCLUSIONS

We proposed a kind of modified local fuzzy weight by use of sensitivity analyses for inner dependence AHP in case of the dependence among alternatives exist. Moreover we can also calculate overall alternative weights for the inner dependence by fuzzy sets.

Our approach shows how to represent weights and how the result of AHP has fuzziness when data is not sufficiently consistent or reliable.

We now plan to investigate the properties of these fuzzy weights more and apply it to real data. In the future work, we will use this idea for not only inner dependence but also outer dependence structure.

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APPENDIX

[Proof] (Theorem 1)

From Perron-Frobenius theorem [11], the Frobenius root λ_A is the simple root. Thus, expansions (6) and (7) are valid. And then, characteristic equations become

$$(A + \varepsilon D_A)(\boldsymbol{w}_1 + \varepsilon \boldsymbol{w}^{(1)} + \boldsymbol{o}(\varepsilon)) \\ = (\lambda_A + \varepsilon\lambda^{(1)} + o(\varepsilon))(\boldsymbol{w}_1 + \varepsilon \boldsymbol{w}^{(1)} + \boldsymbol{o}(\varepsilon)),$$

$$A\boldsymbol{w}_1 = \lambda_A \boldsymbol{w}_1.$$

From these two equations, (9) can be obtained. Further, by Perron-Frobenius theorem, eigenvalue of A and transposed A' is same, therefore

$$\boldsymbol{v}'A = \lambda_A \boldsymbol{v}$$

holds, and it becomes

$$\boldsymbol{v}'\boldsymbol{w}\lambda^{(1)} = \boldsymbol{v}'D_A \boldsymbol{w}.$$

Thus, equation (8) holds. (Q.E.D)

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