First-Order Combinatorics Presenting a Conceptual Framework for Two Levels of Expressive Power of Predicate Logic

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***Abstract*—In this work, we proceed to study finitary and infinitary first-order combinatorics within the framework of a new approach intended to investigations of predicate logic. Some properties of these combinatorics are established. We present a general scheme of semantic layers of model-theoretic properties having importance in the given direction. A number of demonstrations is given showing essence of both finitary and infinitary combinatorial methods for first-order theories. The work represents a basis for further investigations on expressive power of predicate logic.**

***Keywords*—first order logic; theory; finitely axiomatizable theory; computably axiomatizable theory; Tarski-Lindenbaum algebra; model-theoretic property; computation; first-order combinatorics.**

I. Introduction

Constructions of finitely axiomatizable theories were created to answer questions concerning a common problem about expressive power of first-order logic. There are constructions of Church [2], Kleene [7], Hanf [5], Peretyat'kin [12], and others. Each construction represents a general method for constructing finitely axiomatizable theories that can yield a series of finitely axiomatizable theories depending on one or a few input parameters. One can manage properties of the obtained theory by choice of the parameters. Some open questions on expressive power of first-order logic can also be solved with simpler methods based on the signature reduction procedures.

The idea to introduce a first-order combinatorial terminology have arisen from the available approaches to solve a principal problem of characterization of predicate logic of a finite language by building isomorphisms between the Tarski-Lindenbaum algebras of predicate calculi of two finite rich signatures with preservation as a large semantic layer of model-theoretic properties as possible [9][10][11]. The method of constructing such an isomorphism [10] is based on *algorithmic computation* in first-order predicate logic. It uses a universal construction of finitely axiomatizable theories simulating some computation of a Turing machine carrying out the role of a computer-controller. This approach can be said to be an *infinitary first-order combinatorics*. The second method of constructing the isomorphism, [11], is based on a *finite combinatorial transformation* in predicate logic. It uses so-called finite-to-finite signature reduction procedures and can be said to be *finitary first-order combinatorics*. In the work [14], a complex of concepts and general specifications connected with first-order combinatorics was given together with some reasoning justifying the combinatorial terminology for use in this direction. In this work, we introduce a number of further concepts and formulate some claims concerning applications of the finitary and infinitary first-order combinatorial methods for construction and transformation of theories in predicate logic. Besides, a series of general statements is formulated, and a number of demonstrations are given showing essence of the concepts related with finitary and infinitary first-order combinatorics and outlining limits of their possible applications.

In the third section, we introduce definitions of semantic layers relevant in this direction, in the fourth section we introduce a concept of the relation of virtual definable equivalence between theories, the fifth section describes a common scheme of application for infinitary first-order combinatorics, the sixth section specifies possible versions of the universal construction of finitely axiomatizable theories, in the seventh section we list some common statements concerning first-order combinatorics, in the eighth section we describe main situations corresponding to first-order combinatorics. In the ninth section we give some summary to the paper.

II. Preliminaries

Theories in first-order predicate logic with equality are considered. General concepts of model theory, algorithm theory, Boolean algebras, and constructive models can be found in Hodges [6], Rogers [17], and Goncharov and Ershov [4]. Basic concepts concerning first-order combinatorics can be found in [14]. Generally, *incomplete* theories of finite or enumerable signatures are considered.

A finite signature is called *rich*, if it contains at least one $n$ary predicate or function symbol for $n>1$, or two unary function symbols. The following notations are used: $FL(\sigma)$ is the set of all formulas of signature σ, $FL_k(\sigma)$ is the set of all formulas of signature σ with free variables $x_0,\dots,x_{k-1}$, $SL(\sigma)$ is the set of all sentences (i.e., closed formulas) of signature σ. A theory is said to be *computably axiomatizable* if it admits a computable system of axioms. By $L(T)$, we denote the Tarski-Lindenbaum algebra of theory T of formulas without free variables, while $\mathcal{L}(T)$ denotes the Tarski-Lindenbaum algebra $L(T)$ considered together with a Gödel numbering γ; thereby, the concept of a computable isomorphism is applicable to such objects.

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Let σ be a signature, and Σ be a subset of $SL(\sigma)$. By $[\Sigma]^\star$, we denote a theory of a signature $\sigma' \subseteq \sigma$ generated by the set Σ as a set of its axioms, where σ' contains only those symbols from σ that occur in formulas of the set Σ. Let σ^∞ be a fixed enumerable maximally large infinite signature containing countably many both constant symbols, symbols of propositional variables, and predicate and function symbols of each arity $n > 0$. If a theory T of signature σ^∞ is defined by the set of axioms $\{\Phi_i \,|\, i \in W_m\}$ as follows: $T = [\Sigma]^\star$ (where W_m is mth computably enumerable set in Posts's numbering), the number m is called a *weak computably enumerable index* or simply *weak index* of T, and we denote this theory by $T^\star{}_{\{m\}}$, $m \in \mathbb{N}$. This sequence represents all possible computably axiomatizable theories, up to an algebraic isomorphism of theories. Symbol $\mathfrak{P}(X_0, ..., X_\mathfrak{a})$, shortly $\mathfrak{P}$, is specialized to denote a propositional formula of signature $\sigma^* = \{X_0, X_1, ..., X_k, ...; k \in \mathbb{N}\}$ (consisting of propositional variables), while $\mathfrak{a}$ specializes the number of variables occurred in the formula. By *PRO*, we denote the set of all such formulas, while $\mathfrak{P}_i(X_0, ..., X_{\mathfrak{a}(i)})$, $i \in \mathbb{N}$, is a fixed Gödel numbering of the set *PRO*. For a set $A \subseteq \mathbb{N}$, record $A \models \mathfrak{P}$ denotes the value of term $\mathfrak{P}(\chi_A(0), \chi_A(1), ..., \chi_A(\mathfrak{a}))$, where $\chi_A(x)$ is characteristic function of the set A. Here, propositional formula $\mathfrak{P}$ plays the role of a table condition applicable for set $A \subseteq \mathbb{N}$.

Formulation to the *universal construction* $\mathbb{F}\mathbb{U}$ of finitely axiomatizable theories can be found in [12, Ch.6]. Main definitions connected with semantic layers are found in [14]. We use notation *MQL* for the model quasiexact semantic layer presenting *infinitary first-order combinatorics*, [14].

III. First-order combinatorics and a scheme of semantic layers

In accordance with specifications [14], signature reduction procedures represent a basis for the concept of first-order combinatorics; they are considered as particular cases of combinatorial methods in predicate logic. Signature transformations “finite-to-finite” represent finitary combinatorial methods, while signature reduction procedures “infinite-to-finite” represent infinitary combinatorial methods. The problem is to generalize these particular methods to a maximum general approach for which it would be possible to apply such a serious term as ’combinatorics’. A principal aim of the combinatorial approach is to characterize classes of finitary and infinitary methods of transformation of theories. After that, we can define the *finitary semantic layer* as the set of those model-theoretic properties $\mathfrak{p}$ which are preserved under finitary first-order methods, and *infinitary semantic layer* as the set of those properties $\mathfrak{p}$ which are preserved under infinitary methods. For the first-order combinatorial approach, its perfection is considered as a demand of higher priority, while the maximality of the semantic layers of preserved model-theoretic properties is considered as a demand of secondary priority.

In Fig. 1, we present a scheme of inclusions between the semantic layers and similarity relations relevant for first-order combinatorics. Two relations $\approx$ and $\approx_a$ in the top are relations of isomorphism of theories, where $\approx$ means a *model isomorphism* or simply *isomorphism*, while $\approx_a$ means an *algebraic isomorphism* or $\exists\cap\forall$*-presentable equivalence* between two theories. Although $\approx$ and $\approx_a$ are not similarity relations, they are included in the scheme for the sake of completeness. Relations $\equiv_l$ and $\equiv_{al}$ are similarity relations relative to the semantic layer *ML* consisting of all model properties, and respectively, to the layer *AL* consisting of all algebraic properties. Semantic layers *MDL*, *ADL*, *MCL*, etc., and corresponding similarity relations $\equiv_{ad}$, $\equiv_d$, etc., are defined by the classes of *singleton*, *Cartesian*, and respectively, *Cartesian-quotient* interpretations [14]. Leading letter A means an algebraic version while M means a model version. A middle letter S means ’singleton’, C means ’Cartesian’, and

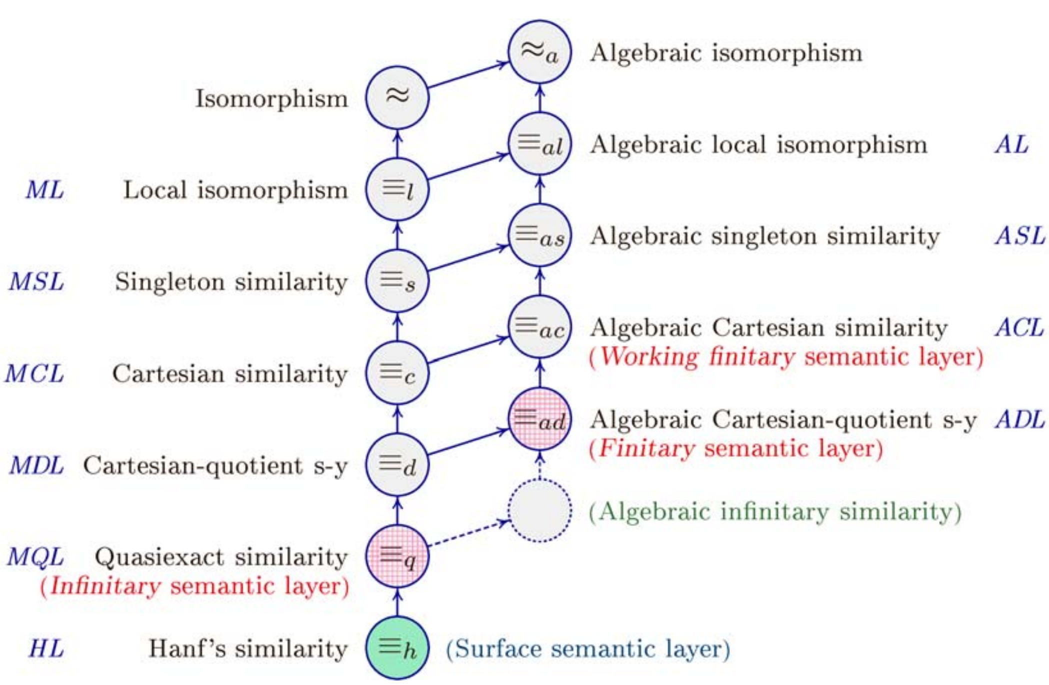

Fig. 1. A scheme of semantic layers of model-theoretic properties

D means 'Cartesian-quotient'. The Hanf layer *HL* is supposed to be $\varnothing$.

Infinitary semantic layer *MQL* [12][14] has a sophisticated definition. Therefore, it would be useful to introduce a simple rule to check whether a model-theoretic property $\mathfrak{p}$ is included in this layer. For this purpose, we consider two following classes of interpretations of theories:

$$I:T\to T\langle\varphi_1^{m_1}/\varepsilon_1,...,\varphi_k^{m_k}/\varepsilon_k\rangle, \tag{1}$$
$$I:T\to T\langle\varphi_1^{m_1}/\varepsilon_1,...,\varphi_k^{m_k}/\varepsilon_k\rangle\oplus SI, \tag{2}$$

where T is a computably axiomatizable theory of an enumerable signature σ, $\varkappa=\langle\varphi_1^{m_1}/\varepsilon_1,...,\varphi_k^{m_k}/\varepsilon_k\rangle$ is a tuple of formulas of signature σ suitable for the Cartesian-quotient extensions, [14, Section 3], while *SI* is the theory of a successor relation with an initial element in signature $\{\lhd^2, c\}$. Based on this, we define two following semantic layers:

`Reference_Block` (3)

`(a)` *FinL* $=$ the set of all model-theoretic properties $\mathfrak{p}$ of algebraic type preserved by any interpretation I of the form (1) with an arbitrary computably axiomatizable theory T and arbitrary tuple $\varkappa$ of this form,

`(b)` *InfL* $=$ the set of all model-theoretic properties $\mathfrak{p}$ of algebraic type preserved by any interpretation I of the form (2) with an arbitrary computably axiomatizable theory T and arbitrary tuple $\varkappa$ of this form.

`End_Ref`

It can be checked that the following assertions take place:

$$\textit{FinL} = \textit{ADL}, \tag{4}$$

$$\textit{InfL} \overset{\mathfrak{R}}{=} I2f\mathfrak{L}, \quad \textit{InfL}\cap \textit{ML} \overset{\mathfrak{R}}{=} I2f\mathfrak{L}\cap \textit{ML} \overset{\mathfrak{R}}{=} Uni\mathfrak{L}. \tag{5}$$

This shows that the family (1) forms a representative class of interpretations for finitary semantic layer *ADL*. On the other hand, a simple modification (2) of the scheme (1) forms a class of interpretations for the semantic layer *InfL*, that, in view of (5), can play the role of a simple rule to check whether a model-theoretic property $\mathfrak{p}$ is included in infinitary semantic layer. The layer *FinL* is said to be the *rapid finitary* semantic layer, while *InfL* is said to be the *rapid infinitary* semantic layer. Simplicity of the definitions (3)(a) and (3)(b) ensures relevance of the layers *FinL* and *InfL* for first-order combinatorics.

As for the algebraic version of the infinitary semantic layer, it is currently not supported by any version of the universal construction of finitely axiomatizable theories. This layer is included in the scheme in Fig. 1 for completeness (shown by a dashed circle).

IV. Virtual definable extensions and finitary first-order combinatorics

There is a known in model theory method of addition to the universe imaginary elements corresponding to a definable set of elements or even to a first-order definable set of tuples of certain length modulo a definable equivalence relation (they are said to be *virtual* elements). Let's add a finite set of virtual regions to the universe. Furthermore, we have to include in signature new predicates distinguishing these areas and establishing a relation of the new elements with the old tuples modulo the equivalence relations. There is a possibility, based on predicate logic, to manipulate with first-order formulas in the extended universe containing the source universe together with the added virtual areas. For this purpose, it is required to define special rules of construction and interpretation of first-order formulas in such an extended region. This method allows us, remaining inside the old universe of models of theory T, to manipulate with language of first-order logic in models of some new theory S, which is possible said to be a *virtual first-order definable extension* of the source theory T. One can mention that, any model-theoretic properties of theories T and S should be considered as coincided since the virtually extended theory S is presented in the initial theory T. Furthermore, notice that such an operation of addition of a finite number of virtual definable regions can be performed in a general situation when the source theory T is incomplete. In this case, we obtain a computable isomorphism between the Tarski-Lindenbaum algebras $\mu:\mathcal{L}(T)\to\mathcal{L}(S)$ preserving all really model-theoretic properties.

Two theories T and S are said to be *virtually definably equivalent*, written as $T\approxeq S$, if there are virtual definable extensions T' of T and S' of S such that T' and S' are algebraically isomorphic, $T'\approx_a S'$. This relation, close to that considered in Manders [8], seems to be the most common equivalence relation between first-order theories. Since the operation of a virtual definable extension of a theory is closely related to the operation of Cartesian-quotient extension of a theory, [14, Section 3], we obtain that this relation between theories plays the principal role within the complex of concepts for finitary first-order combinatorics.

V. Normalized scheme for infinitary first-order combinatorics

In this section, we specify some method of construction of finitely axiomatizable theories with pre-assigned model-theoretic properties. In the most common case, the target theory depends on an input parameter n. Our goal is to construct a finitely axiomatizable theory $F=F^{(n)}$ of a given finite rich signature τ. First, we build an intermediate computably axiomatizable theory $T=T^{(n)}$ using some particular method. Signature of T:

$$\sigma=\{X_i\mid i\in\mathbb{N}\}\cup\sigma', \tag{6}$$

where X_i, $i\in\mathbb{N}$, is a sequence of nulary predicates (i.e., propositional variables), and σ' depends on the aim of our construction. Axioms of T consist of three groups:

Frame: a group of axioms describing general form of a so-called skeleton of the theory; these axioms depend on the aims of the construction;

Space: formulas of the form $\mathfrak{P}(X_0,...,X_\mathfrak{a})$, with $\mathfrak{P}\in \textit{PRO}$;

Ext: formulas of the form $\mathfrak{P}(X_0,...,X_\mathfrak{a})\to\Psi$, with $\mathfrak{P}\in \textit{PRO}$ and $\Psi\in SL(\sigma')$.

Applying the universal construction $\mathbb{F}\mathbb{U}$, we build a finitely axiomatizable theory $F=F^{(n)}=\mathbb{F}\mathbb{U}(T,\tau)$ of the wished finite rich signature τ together with a computable isomorphism $\mu: \mathcal{L}(T)\to\mathcal{L}(F)$ between the Tarski-Lindenbaum algebras preserving model-theoretic properties of their completions within the infinitary semantic layer *MQL*.

Introduce the following notation

$$\theta_i = \mu(X_i),\ i \in \mathbb{N}. \tag{7}$$

For an arbitrary set $A \subseteq \mathbb{N}$, we denote

$$\begin{aligned} T[A] &= T + \{X_i \mid i \in A\} \cup \{\neg X_j \mid j \in \mathbb{N} \setminus A\}, \\ F[A] &= F + \{\theta_i \mid i \in A\} \cup \{\neg \theta_j \mid j \in \mathbb{N} \setminus A\}. \end{aligned} \tag{8}$$

Furthermore, we define a number m such that

$$W_m = \{k \mid T \vdash \mathfrak{P}_k(X_0,...,X_{\mathfrak{a}(k)})\}, \tag{9}$$

and introduce the following notation

$$\Omega(m) = \{A \subseteq \mathbb{N} \mid (\forall k \in \Omega(m)) A \models \mathfrak{P}_k\}. \tag{10}$$

Any object involved in the transformation $n \mapsto T \mapsto F$ is presented via appropriate computably enumerable index or Gödel number such that the whole passage $n \mapsto T \mapsto F$ is defined by an effective operator relative to indices and/or Gödel numbers. The given complex of transformations is said to be *normalized* if the following conditions are satisfied:

$$\begin{aligned} &\text{(a)}\ T \vdash \mathfrak{P}(X_0,...,X_{\mathfrak{a}}) \Leftrightarrow T.Spase \vdash \mathfrak{P}(X_0,...,X_{\mathfrak{a}}), \mathfrak{P} \in \textit{PRO}, \\ &\text{(b)}\ (\forall A \subseteq \mathbb{N})\ T[A] \textit{ is either complete or contradictory.} \end{aligned} \tag{11}$$

These conditions are, in fact, natural. If we have an arbitrary effective transformation $n \mapsto T' \mapsto F'$, where T' is a computably axiomatizable theory constructed from n, and F' is a finitely axiomatizable theory of signature τ obtained from T' by the universal construction, then, this scheme can equivalently be transformed in the form of a normalized scheme $n \mapsto T \mapsto F$. Furthermore, any normalized complex must satisfy the following properties: (a) $T[A]$, $A \in \Omega(m)$ represents the family of all complete extensions of T; (b) $F[A]$, $A \in \Omega(m)$, represents the family of all complete extensions of F; (c) isomorphism μ maps $T[A]$ to $F[A]$, for all $A \in \Omega(m)$; (d) for any $A \in \Omega(m)$ complete theories $T[A]$ and $F[A]$ have identical model-theoretic properties within the infinitary semantic layer *MQL*; (e) effectively, in the system of axioms of T, one can find $s \in \mathbb{N}$ such that function $\varphi_s^A(t)$ is characteristic for the set $\mathrm{Nom}(T[A])$, for all $A \in \Omega(m)$; a definition of $\varphi_s^A(t)$ is found in [17, p.130].

VI. VERSIONS OF THE UNIVERSAL CONSTRUCTION

Hereafter, we use notation $\mathcal{MQL}$ for a sublayer of the infinitary layer *MQL*.

Simplest form of the universal construction, denoted $\mathbb{F}\breve{\mathbb{U}}$, is presented by:

Statement 1. [GENERIC UNIVERSAL CONSTRUCTION: A PRIMITIVE FORM] *The following assertion holds* (*where* $\mathcal{MQL} \subseteq$ *MQL*):

$$(\forall\ c.a.\ theory\ T)(\exists\ f.a.\ theory\ F)\left[T \equiv_{\mathcal{MQL}} F\right]. \tag{12}$$

A more common formulation to the universal construction, [12, Th.0.6.1]:

Statement 2. [GENERIC UNIVERSAL CONSTRUCTION: A NORMAL FORM] *Given an arbitrary computably axiomatizable theory T and a finite rich signature σ. Effectively in a weak c.e. index of T and Gödel number of σ, one can construct a finitely axiomatizable theory $F = \mathbb{F}\mathbb{U}(T,\sigma)$ of signature σ together with a computable isomorphism $\mu : \mathcal{L}(T) \to \mathcal{L}(F)$ between the Tarski-Lindenbaum algebras preserving all model-theoretic properties within the layer $\mathcal{MQL}$* (*it is supposed that* $\mathcal{MQL} \subseteq$ *MQL*).

The following dependence statement takes place.

Lemma 3. *Having any version of the universal construction in the primitive form* (12) *with the semantic layer $\mathcal{MQL} \subseteq$ MQL, one can restore the missing effectiveness requirement obtaining its normal form presented in Statement* 2 *with the same layer $\mathcal{MQL}$.*

PROOF. First, we introduce an operation with a sequence of theories. We use sequence $T^{\star}{}_{\{n\}}$, $n \in \mathbb{N}$, including all, up to an isomorphism, c.a. theories, cf. Preliminaries. Let $T^{\star}{}_{\{n\}}$ has signature σ_n. It is assumed that $\sigma_n \cap \sigma_k = \varnothing$ for all n, k such that $n \neq k$. Consider the following new signature

$$\sigma' = \{Z_i^0 \mid i \in \mathbb{N}\} \cup \{U^1, c\} \cup \sigma_0 \cup \sigma_1 \cup \ldots \cup \sigma_k \cup \ldots, \tag{13}$$

where Z_i^0, $i \in \mathbb{N}$, are symbols of nulary predicates. It is assumed that the symbols U, c, and Z_i, $i \in \mathbb{N}$, do not belong to $\sigma_0 \cup \sigma_1 \cup \ldots \cup \sigma_k \cup \ldots$.

Construct theory $T^u_{c.a.}$ of signature σ' defined by the following set of axioms:

1°. $U(x) \leftrightarrow (x \neq c)$,

2°. $(\exists x) U(x)$,

3°. $Z_n \to \neg Z_k$, $n, k \in \mathbb{N}$, $n \neq k$,

4°. $Z_n \to$ (*on* $U(x)$, *axioms of* T_n *are satisfied*), $n \in \mathbb{N}$,

5°. $Z_n \to$ (*outside* $U(x)$, σ_n*-symbols defined trivially*),

6°. $\neg Z_k \to$ (*all* σ_k*-symbols are defined* c*-trivially*), $k \in \mathbb{N}$.

Denote this theory by $\bigotimes_{n \in \mathbb{N}} T^{\star}{}_{\{n\}}$. The statement above "defined c-trivially" means that all σ_k-predicates are identically false, each σ_k-function f^m satisfies $f(x_1,...,x_m) = x_1$ for all its arguments, and each σ_k-constant is interpreted by c.

We can show that the following assertions hold:

(a) theory $T^u_{c.a.} = \bigotimes_{n \in \mathbb{N}} T_n$ is computably axiomatizable;

(b) for any $n \in \mathbb{N}$, theory $T^u_{c.a.} \cup \{Z_n\}$ is algebraically isomorphic to the constant extension $T^{\star}{}_{\{n\}}\langle c\rangle$ of the theory $T^{\star}{}_{\{n\}}$;

(c) there is a computable isomorphism $\mu_n : \mathcal{L}(T^{\star}{}_{\{n\}}) \to \mathcal{L}(T^u_{c.a.} \cup \{Z_n\})$ preserving all model-theoretic properties within the semantic layer *ASL*.

Part (a) is a consequence of the fact that the sequence $T^{\star}{}_{\{n\}}$, $n \in \mathbb{N}$, is computable. Part (b) is checked immediately. Part (c) is a consequence of (b).

Now, we are going to use the universal c.a. theory $T^u_{c.a.}$ to deduce the normal form of the universal construction from its primitive form. Applying the primitive form (12) of the universal construction, we find a finitely axiomatizable theory F_0 together with a computable isomorphism $\mu_0 : \mathcal{L}(T^u_{c.a.}) \to \mathcal{L}(F_0)$ preserving the layer $\mathcal{MQL}$. After that, a construction with the effectiveness requirement is obtained as an immediate consequence of the universality condition for $T^u_{c.a.}$ stated in (a)–(c); namely, we have to perform the following transformation:

$$T^{\star}{}_{\{n\}} \mapsto \underbrace{T^{u}_{c.a.}+\{Z_n\}}_{S} \mapsto \mathbb{F}\check{\mathbb{U}}(S) \mapsto \mathrm{Redu}(\mathbb{F}\check{\mathbb{U}}(S),\sigma), \quad (14)$$

where $\mathrm{Redu}(H,\sigma)$ denotes a signature reduction procedure from a finitely axiomatizable theory H to such a theory of finite rich signature σ. By construction, we can effectively build a computable isomorphism between the Tarski-Lindenbaum algebras of theories $T^{\star}{}_{\{n\}}$ and $\mathrm{Redu}(\mathbb{F}\check{\mathbb{U}}(T^{u}_{c.a.}+\{Z_n\}),\sigma)$. Thus, the transformation (14) can play the role of a normal form of the universal construction with the layer *MQL*. ☐

The following statement represents so-called *universal-under-canonical* construction; alternatively, it is said to be the *canonical-mini* construction:

Statement 4. *There is a routine proof* (*by way of transformation of theories based on the methods of infinitary first-order combinatorics*) *that, from statement of the canonical construction,* [*12, Ch.3, Th.3.1.1*], *deduces a weak version of the universal construction with the following semantic layer of model-theoretic properties* (*denoted by* $M1L^{\circ}$):

(a) *existence of a prime model, its strong constructivizability, and the value of its algorithmic dimension* (*relative to strong constructivizations*);
(b) *existence of a countable saturated model and its strong constructivizability*.

PROOF. Only outline of the proof is given. The canonical construction can control those model-theoretic properties which are expressible in signature $\sigma^{*}=\{P^1_0,P^1_1,...,P^1_k,...;k\in\mathbb{N}\}$ with infinitely many unary predicates (pay an attention: Chapter 3 of [12] is titled "The construction over a unary list", where the *list* means a *layer*). On the other hand, the pointed out layer $M1L^{\circ}$ consists of exactly those model-theoretic properties controlled by the canonical construction, which are expressible in terms of structure of the Tarski-Lindenbaum algebras $\mathcal{L}_n(T)$, $n\in\mathbb{N}$, $n>0$, of theory T.

Let T be an arbitrary computably axiomatizable theory of an enumerable signature σ' having Gödel numbering Φ_i, $i\in\mathbb{N}$, for the set $SL(\sigma')$. The sequence of sentences Φ_i, $i\in\mathbb{N}$, is a generating set for the Tarski-Lindenbaum algebra $\mathcal{L}(T)$. Enrich σ' with propositional variables X_i, $i\in\mathbb{N}$, and add to T additional axioms $X_i\leftrightarrow\Phi_i$, $i\in\mathbb{N}$. Construct parameterized Stone space $\Omega(m)$ for T relative to generating sequence X_i, $i\in\mathbb{N}$. Considering a set $A\in\Omega(m)$ as an oracle, let's construct Boolean algebra $\mathcal{B}=\bigotimes_{0<i<\omega}\mathcal{L}_k(T[A])$ that, in fact, is a c.e. Boolean algebra relative to computation with oracle A. It is an important moment that satisfaction of each model-theoretic property $\mathfrak{p}\in M1L^{\circ}$ in theory T is expressible (in a known way) via the algebra $\mathcal{B}$. On the other hand, we can present the algebra $\mathcal{B}$ (depending on oracle A) via c.e. binary tree computable with the same oracle. This gives a value to the second parameter $s\in\mathbb{N}$ to the canonical construction. Applying the construction $\mathbb{F}\mathbb{c}$ to the obtained pair of input arguments (m,s), we finally build theory $F=\mathbb{F}\mathbb{c}(m,s,\sigma)$ that, by virtue of main statement of the canonical construction, is the required finitely axiomatizable theory. ☐

Mention that, an available proof for the canonical construction is essentially simpler in comparison with that for the universal construction. On the other hand, the proof given above represents a demonstration of the methods of infinitary first-order combinatorics.

VII. Summary: some common statements concerning first-order combinatorics

In this paragraph, we formulate a series of common statements corresponding to finitary and infinitary first-order combinatorics (or not corresponding to such a combinatorics).

S1. In the case of finitary first-order combinatorics, characteristic property of the transformation between theories is availability of a one-to-one mapping between the isomorphism types of their models (this property is said to be the *model-bijectiveness*).

S2. In the case of infinitary first-order combinatorics, a characteristic property of the construction is availability of non-standard fragments in models of the target theory, whose description should be simple enough; moreover, this simplicity is a principal demand of infinitary first-order combinatorics.

S3. In the case of infinitary first-order combinatorics, our goal is to build a computably axiomatizable theory that, generally, may be incomplete; a description of the family of all complete extensions of the theory should be presented; the axioms should provide some pre-assigned properties of these extensions depending on an input parameter; applying an appropriate version of the universal construction, we obtain the target finitely axiomatizable theory.

S4. In the case of infinitary first-order combinatorics, the input parameter could be absent if our goal is to build a separate example of finitely axiomatizable theory with some pre-assigned properties; on the other hand, we can use a few input parameters (more than one) if it is necessary for the problem considered.

S5. In the case of infinitary first-order combinatorics, a complete theory may be considered as a particular case of incomplete theories; however, if the purpose is limited with complete theories only, such a construction does not correspond to specifications of infinitary first-order combinatorics or is weakly linked with it.

S6. If we consider or build an incomplete theory, but it is impossible to parameterize the family of its complete extensions, such a construction does not correspond to specifications of infinitary first-order combinatorics or is weakly linked with it.

S7. In the case of infinitary first-order combinatorics, first of all, the used methods of construction or transformation of theories are principal; as for the requirements of computability of the construction and enumerability of the signature, they ordinarily are satisfied automatically.

S8. In the case of infinitary first-order combinatorics, a sublayer of the full infinitary semantic layer *MQL* may be considered; an empty layer $\varnothing$ is also admissible.

VIII. Demonstrations: situations corresponding to first-order combinatorics

In this section, we consider a number of typical examples of applications of finitary and infinitary combinatorial methods; we also demonstrate some situations when methods of construction and transformation of theories does not correspond to the concept of first-order combinatorics.

1) Definitionally equivalent theories: In [16, p. 481], C. Pinter writes, "There are many common instances of theories, which may be formulated naturally in more than one way, using different sets of primitive relations and operations. For example, lattice theory may be presented as a ... theory ... with the operations $+$ and $\cdot$, or alternatively, as a theory ... whose language has only one nonlogical symbol $\leqslant$. ... When theories T and T' are related in this manner, they are said to be *definitionally equivalent*." Similar sense has the concept of relation of *synonymity of theories* introduced in Bouvere [1]. As mentioned in [18, p.130], "... synonymity requires the universe to remain unchanged," the same is also true relative to Pinter's definitional equivalence. Some additional examples: Boolean algebras can be considered in the signature either $\{\cup,\cap,-,0,1\}$, or $\{\subseteq,0,1\}$, or even $\{+,\cdot,0,1\}$; group theory can be considered in the signature either $\{\cdot,e\}$, or $\{\cdot\}$, or even $\{+,\theta\}$, etc. In these situations, we have a simplified version of finitary first-order combinatorics (because more common virtual definable extensions of theories are not used here).

2) Virtual definitionally equivalent theories: Some situations which are close to finitary first-order combinatorics were discussed by Leslaw Szczerba in [18]. At [18, p. 130] he writes, "... authors frequently use sequences of elements as new elements, members of a new universe (e.g., points may be pairs of real numbers as in the case of the Cartesian plane), universes might be restricted to definable subsets, and moreover, new elements might be equivalence classes with respect to some definable equivalence relation." These statements exactly correspond to the concept of Cartesian extension of a theory and, in other words, to the concept of a virtual definitional extension of a theory. Here, we exactly have finitary first-order combinatorics. Dale Myers in [9, p.85] calls this transformation an *interpretive isomorphism* and states that this definition was introduced in Manders [8] (author's remark: Manders' definition is based on Szczerba's ideas). Some additional examples: (a) Consider the class K of models, which are Boolean algebras in signature $\{\cup,\cap\}$ with omitted both particular elements 0 and 1; thereby, the operations are partial. Alternatively, we can consider this class of systems in signature $\{\subseteq\}$. Applying virtual definable extension to $\mathrm{Th}(K)$, we can obtain theory *BA* of Boolean algebras. From this fact, we obtain that theories $\mathrm{Th}(K)$ and *BA* have identical model-theoretic properties (namely, there is a computable isomorphism $\mu:\mathcal{L}(\mathrm{Th}(K))\to\mathcal{L}(BA)$ that preserves all model-theoretic properties). (b) Another example is a system of positive real numbers $\mathfrak{N}=(\mathbb{R}^{>0},\cdot,+,-)$ with partial operation $-$. Applying virtual definable extension to $\mathrm{Th}(\mathfrak{N})$, we can obtain theory $\mathrm{Th}(\mathfrak{M})$, where $\mathfrak{M}=(\mathbb{R},\cdot,+,-)$; thereby, theories $\mathrm{Th}(\mathfrak{N})$ and $\mathrm{Th}(\mathfrak{M})$ have identical model-theoretic properties.

In all these cases, we have a situation of finitary first-order combinatorics.

3) Particular examples of finitely axiomatizable theories: Suppose that we are going to construct a finitely axiomatizable theory F of a given finite rich signature σ satisfying the following properties: the set of all complete extensions of F consists of a countable sequence F_k, $k\in\mathbb{N}\cup\{\omega\}$, such that, each of the theories F_0, F_1, F_2, ... is ω-stable theory and is finitely axiomatizable over F, while F_ω is not finitely axiomatizable over F, it is not ω-stable and has a prime model. First, we have to find a computably axiomatizable theory T with these properties (it is a simple exercise to build such a theory). Applying the universal construction to T, we can pass a finitely axiomatizable theory $F=\mathbb{F}\mathbb{U}(T,\sigma)$ together with a computable isomorphism $\mu:\mathcal{L}(T)\to\mathcal{L}(F)$ preserving all model-theoretic properties of the infinitary semantic layer *MQL*. Because both properties $\mathfrak{p}_1=$"theory is ω-stable" and $\mathfrak{p}_2=$"theory has a prime model" belong to *MQL*, we obtain finally that the theory F indeed satisfies the posed properties.

This example demonstrates methods of infinitary first-order combinatorics.

4) Algorithmic complexity estimates for semantic classes: Let σ be a finite rich signature, and Φ_i, $i\in\mathbb{N}$, be a fixed Gödel numbering for the set of sentences of this signature. We are going to prove the following statement.

Theorem 5. $\{n\,|\,\Phi_n$ *determines a complete theory*$\}\approx\Pi^0_2$.

PROOF. The upper estimate can be established immediately.

For the lower estimate, we consider the following m-universal in Π^0_2 set: $I=\{n\,|\,W_n$ *is infinite* $\}$, [17, Th.13-VIII, p.264]. Signature of the theory $\sigma=\{X_0,...,X_i,...\}$ consists of propositional variables (i.e., nulary predicates). Given an input parameter n. Consider computably axiomatizable theory $T=T^{(n)}$ of signature σ, determined by the following set of axioms:

1°. $X_k\leftrightarrow(\exists x_1...x_k)\bigwedge_{0<i<j\leqslant k}(x_i\neq x_j)$,

2°. X_k, $k\in W_n$.

Applying the universal construction $\mathbb{F}\mathbb{U}$ to $T^{(n)}$, we effectively find a finitely axiomatizable theory $F=F^{(n)}=\mathbb{F}\mathbb{U}(T^{(n)},\sigma)$ of signature σ together with a computable isomorphism $\mu:\mathcal{L}(T)\to\mathcal{L}(F)$. First, consider the case $n\in W_n$. In this case, W_n is infinite, so all models of T are infinite and the theory is ω_0-categorical; thus, T is complete by Vaught Theorem. Now, consider the case $n\notin W_n$. In this case, W_n is finite, thereby, theory T cannot be complete since it has both finite and infinite models. As a result, we have obtained that the theory T is complete if and only if $n\in I$; thus, we have

$$n\in I\Leftrightarrow T_n \text{ is complete } \Leftrightarrow F_n \text{ is complete.}$$

The theory $F^{(n)}$ is defined effectively in $T^{(n)}$. Therefore, there exists a total computable function $f(n)$ such that the sentence $\Phi_{f(n)}$ is an axiom of this theory. Finally, we obtain the required lower estimate:

$$n\in I\Leftrightarrow\Phi_{f(n)} \text{ determines a complete theory.}$$

Proof of Theorem 5 demonstrates methods of infinitary first-order combinatorics. Furthermore, there are lots of results in this direction in [12, Ch. 8].

5) Isomorphisms between predicate calculi of different finite rich signatures: We consider a problematic concerning the isomorphism type of the Tarski-Lindenbaum algebra $\mathcal{L}(PC(\sigma))$ of predicate calculus $PC(\sigma)$ of a finite rich signature σ. Methods of [11] determine a Hanf's isomorphism μ between the Tarski-Lindenbaum algebras $\mathcal{L}(PC(\sigma_1))$ and $\mathcal{L}(PC(\sigma_2))$ of any two finite rich signatures σ_1 and σ_2. It is assembled from a countable set of partial mappings, which are finite-to-finite signature reduction procedures; thereby, this isomorphism μ preserves all really model-theoretic properties. The work by Myers [9] also defines such an isomorphism between the predicate calculi $\mathcal{L}(PC(\sigma_1))$ and $\mathcal{L}(PC(\sigma_2))$ that is assembled from partial mappings described by Gaifman's maps [3]. These two approaches coincide with each other from the point of view of finitary first-order combinatorics.

6) Structure of the Tarski-Lindenbaum algebras of semantic classes: There are lots of results in this direction in papers [13][15] and others. Their proofs demonstrate methods of infinitary first-order combinatorics.

7) Definability in Peano Arithmetic and set theory: Let T be a rich theory like arithmetic or set theory (for definiteness, let T be Peano Arithmetic). It is a known fact that T is not complete; moreover, any finitely axiomatizable extension of T cannot be complete. Thereby, we have obtained that the Tarski-Lindenbaum algebra $\mathcal{L}(T)$ must be countable, atomless Boolean algebra. However, axiomatic of the theory T is such that neither direct parameterization nor even understandable description of the family of all complete extensions is possible. Thus, argumentation of statement S6, cf. Section VII, is applicable to this situation concerned to rich formal systems. Some additional examples: Church construction [2], Kleene construction [7] (presenting an extension of Peano Arithmetic), first-order presentation of any universal computing system, etc.

All these examples demonstrate situations outside of the approach based on the methods of first-order combinatorics.

IX. Conclusion

The work presents some extra details and gives general demonstrations and specifications to the concepts of finitary and infinitary first-order combinatorics. Based on both formal substantiations and informal arguments, we show that the introduced complex of concepts and definitions for the first-order combinatorics adequately corresponds to the problems on expressive possibilities of predicate logic presenting a firm basis for Computer Science as well as for other branches of mathematics.

In Section IV, we introduced the concept of virtual definable equivalence between theories; this relation presents essence of finitary first-order combinatorics. Further, in Section V, we describe a scheme of application of infinitary first-order combinatorics. This scheme represents the most general form of a computable procedure to build a theory T from a complex C of objects of computational nature with a transformation of the obtained theory T to a finitely axiomatizable theory F together with a computable isomorphism $\mu : \mathcal{L}(T) \to \mathcal{L}(F)$ between their Tarski-Lindenbaum algebras preserving model-theoretic properties within the infinitary semantic layer *MQL* whose fundamental nature is established in [14]. In fact, a special set $X \subseteq \mathbb{N}$ is used in this construction presenting a parameterization for Stone space of the target theory. This set X plays the role of an oracle; thereby, the transformation related to infinitary first-order combinatorics represents, as a whole, is a common Turing computation (it is possible to say, computable Brute Force with an oracle).

Summarizing, we can say that, the combinatorial approach requires sophisticated definitions and is partially based on informal substantiation. Nevertheless, the concepts of finitary and infinitary first-order combinatorics adequately correspond to the posed class of problems; moreover, the informal argumentations and limitations are rather natural justifying the appropriateness of using the combinatorial terminology in this direction.

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