

Incremental Reasoning on Strongly Distributed Fuzzy Systems

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Abstract—We introduce the notion of strongly distributed fuzzy systems and present a uniform approach to incremental problem solving on them. The approach is based on the systematic use of two logical reduction techniques: Feferman-Vaught reductions and syntactically defined translation schemes. The fuzzy systems are presented as logical structures \mathcal{A} 's. The problems are presented as fuzzy formulae on them. We propose a uniform template for methods, which allow (for a certain cost) evaluation of formulae of fuzzy logic \mathcal{L} over \mathcal{A} from values of formulae over its components and values of formulae over the index structure \mathcal{I} .

Keywords—Fuzzy systems; Incremental reasoning; Reduction sequences; Syntactically defined translation schemes; Translations; Transductions.

I. INTRODUCTION

Decomposition and incremental reasoning on switch systems comes back to early 60's [1] [2]. Since Zadeh introduced the fuzzy set theory in [3], by exploiting the concept of membership grade, numerous attempts to investigate fuzzy systems and their properties have been applied. In this context, the theory of disjunctive decompositions of [1] [2] and others, was shown to be insufficient. From the pioneering works, investigating the problem, we mention only [4], where an approach for obtaining simple disjunctive decompositions of fuzzy functions is described. However, the approach is not scalable to large functions and hardly implemented. The summary of the first results may be found in [5]. See [6] for the next contributions in the field.

Jumping to the 90's, we mention [7], which deals with the problem of general max-min decomposition of binary fuzzy relations defined in the Cartesian product of finite spaces. In [8], a new method to derive the membership functions and reference rules of a fuzzy system was developed. Using this method, a complicated Multiple Input Single Output system can be obtained from combination of several Single Input Single Output systems with a special coupling method. Moreover, it was shown how the decomposition and coupling method reduces complexity of the network, used to represent the fuzzy system. Theoretical results on structural decomposition of general Multiple Input Multiple Output fuzzy systems are presented in [9]. Some recent results on a decomposition technique for complex systems into hierarchical and multi-layered fuzzy logic sub-systems may be found in [10].

For α -decomposition of [11], originated from *max* – *min* composition of [12], in [13], it was shown that every fuzzy relation R is always generally effectively α -decomposable.

Moreover, calculating of $\rho(R) = \min\{|Z| : R = Q\alpha T, Q \in F(X \times Z), T \in F(Z \times Y)\}$ is an NP-complete problem. A new concept for the decomposition of fuzzy numbers into a finite number of α -cuts provided in [14].

In this paper, we propose a *generalized* purely theoretical approach to incremental reasoning on fuzzy distributed systems. This approach allows us to give the precise definition of locality. Moreover, we propose a template, such that if one follows it successfully then it is possible to reduce evaluation of a fuzzy formula on the composed structure to evaluation of *effectively algorithmically* derived formulae on components with a final post-processing. We show two cases, when the template may be successfully applied to two the most popular semantics of fuzzy logic. Finally, we give some complexity analysis of the method.

We consider fuzzy logic as a infinite-valued (infinitely-many valued) logic, in which the law of excluded middle does not hold. In fact, the truth function for an extension of a First Order Logic (*FOL*) relation R with a fuzzy relation is a mapping in the interval $[0, 1]$. History of many-valued logics (a propositional calculus, in which there are more than two truth values) comes back to early 20's of the previous century [15][16]. One of the first formalizations of such a view may be found in [17]. The approach leads to the following definition of a fuzzy truth-value lattice [18]: A *fuzzy truth-value lattice* is a lattice of fuzzy sets on $[0, 1]$ that includes two complete sublattices \mathbf{T} and \mathbf{F} such that:

- 1) $\forall v_1 \in \mathbf{T} \forall v_2 \in \mathbf{F} : v_1$ and v_2 incomparable, and
- 2) $\forall S \in \mathbf{T} : \text{lub}(S) \in \mathbf{T}$ and $\text{glb}(S) \in \mathbf{T}$, moreover $\forall S \in \mathbf{F} : \text{lub}(S) \in \mathbf{F}$ and $\text{glb}(S) \in \mathbf{F}$, and
- 3) $\forall v \in \mathbf{T} \forall \epsilon \in [0, 1] : \text{if } \exists v^* \in \mathbf{T} : v^* \leq_l v + \epsilon$ then $v + \epsilon \in \mathbf{T}$, moreover $\forall v \in \mathbf{F} \forall \epsilon \in [0, 1] : \text{if } \exists v^* \in \mathbf{F} : v^* \leq_l v + \epsilon$ then $v + \epsilon \in \mathbf{F}$, where

\mathbf{T} and \mathbf{F} respectively denote the set of all **TRUE**-characteristic truth-values and the set of all **FALSE**-characteristic false-values in the lattice; *lub* and *glb* are the labels of the *least upper bound* and the *greatest lower bound*.

In a particular definition of a truth-value lattice, *lub* and *glb* are interpreted by specific operations. There exists a variety of fuzzy set intersection and union definitions [19], and *lub* and *glb* can be defined to be any corresponding ones of them. Moreover, systems based on real numbers in $[0, 1]$ having truth-characteristics distinguished [17], commonly use 0.5 as the splitting point between **FALSE**- and **TRUE**-characteristic

regions, where 0.5 is considered an **UNKNOWN**- characteristic truth-value. In such a case, *lub* corresponds to *max*, *glb* corresponds to *min* and \leq is the usual real number less-than-or-equal-to relation. For a possible connection of fuzzy logic and graph grammars, see [20].

In this paper, we generalize and extend the coupling method of [8] by systematic application of two logical reduction techniques to the field of reasoning on fuzzy distributed systems. The distributed systems are presented as logical structures \mathcal{A} 's. We propose a uniform template for methods, which allow for certain cost evaluation of formulae of fuzzy logic \mathcal{L} (with a particular choice of *lub* and *glb*) over \mathcal{A} from values of formulae over its components and values of formulae over the index structure \mathcal{I} . In this paper, we consider only relational structures. We assume that the reader has general logical background as may be found in [21], [22].

The logical reduction techniques are:

Feferman-Vaught reduction sequences (or simply, reductions) were introduced in [23]. Given logical structure \mathcal{A} as a composition of structures $\mathcal{A}_i, i \in I$ and index structure \mathcal{I} . Reduction sequence is a set of formulae such that each such a formula can be evaluated locally in some site or index set. Next, from the local answers, received from the sites, and possibly some additional information about the sites, we compute the result for the given global formula. In the logical context, the reductions are applied to a relational structure \mathcal{A} distributed over different sites with structures $\mathcal{A}_i, i \in I$. The reductions allow the formulae over \mathcal{A} to be computed from formulae over the \mathcal{A}_i 's and formulae over index structure \mathcal{I} .

Translation schemes are the logical analogue to coordinate transformations in geometry. The fundamental property of translation schemes describes how to compute transformed formulae in the same way Leibniz' Theorem describes how to compute transformed integrals. The fundamental property has a long history, but was first properly stated by Rabin [24].

General complexity analysis of incremental computations in the proposed framework may be found in [25].

The paper is organized as follows. In Section II, we discuss different ways of obtaining structures from components. Section III introduces the notion of abstract translation schemes. Section IV is the main section of the paper, where we state and prove our main Theorem 4. Section VI summarizes the paper.

II. DISJOINT UNION AND SHUFFLING OF STRUCTURES

The first reduction technique that we use is *Feferman-Vaught reductions* [23]. In this section, we start to discuss different ways of obtaining structures from components. We mostly follow [26][27]. The *Disjoint Union* of a family of structures is the simplest example of juxtaposing structures over an index structure \mathcal{I} with universe I , where none of the components are linked to each other. In such a case the index structure \mathcal{I} may be replaced by an index set I .

We start our considerations from First Order Logic (*FOL*). Second Order Logic (*SOL*) is like *FOL* but allows quantification over relations. If the arity of the relation restricted to 1 then we deal with Monadic Second Order Logic (*MSOL*). We recall the following definitions:

Definition 1 (Quantifier Rank of Formulae): Quantifier rank of formula φ ($qr(\varphi)$) is defined as follows:

- for φ without quantifiers $qr(\varphi) = 0$;
- if $\varphi = \neg\varphi_1$ and $qr(\varphi_1) = n_1$, then $qr(\varphi) = n_1$;
- if $\varphi = \varphi_1 \cdot \varphi_2$, where $\cdot \in \{\vee, \wedge, \rightarrow\}$, and $qr(\varphi_1) = n_1$, $qr(\varphi_2) = n_2$, then $qr(\varphi) = \max\{n_1, n_2\}$;
- if $\varphi = Q\varphi_1$, where Q is a quantifier, and $qr(\varphi_1) = n_1$, then $qr(\varphi) = n_1 + 1$.

Definition 2 (Disjoint Union):

Let $\tau_i = \langle R_1^i, \dots, R_{j_i}^i \rangle$ be a vocabulary of structure \mathcal{A}_i . In the general case, the resulting structure is $\mathcal{A} = \bigsqcup_{i \in I} \mathcal{A}_i = \langle I \cup \bigcup_{i \in I} \mathcal{A}_i, P(i, v), Index(x), R_j^I (1 \leq j \leq j^I), R_{j_i}^i (i \in I, 1 \leq j^i \leq j^I) \rangle$ for all $i \in I$, where $P(i, v)$ is true iff element a came from \mathcal{A}_i , $Index(x)$ is true iff x came from I .

Definition 3 (Partitioned Index Structure):

Let \mathcal{I} be an index structure over τ_{ind} . \mathcal{I} is called *finitely partitioned* into ℓ parts if there are unary predicates $I_\alpha, \alpha < \ell$, in the vocabulary τ_{ind} of \mathcal{I} such that their interpretation forms a partition of the universe of \mathcal{I} .

The following classical theorem holds:

Theorem 1:

Let \mathcal{I} be a finitely partitioned index structure. Let $\mathcal{A} = \bigsqcup_{i \in I} \mathcal{A}_i$ be a τ -structure, where each \mathcal{A}_i is isomorphic to some $\mathcal{B}_1, \dots, \mathcal{B}_\ell$ over the vocabularies τ_1, \dots, τ_ℓ , in accordance to the partition (ℓ is the number of the classes). For every $\phi \in MSOL(\tau)$ there are:

- a boolean function $F_\phi(b_{1,1}, \dots, b_{1,j_1}, \dots, b_{\ell,1}, \dots, b_{\ell,j_\ell}, b_{I,1}, \dots, b_{I,j_I})$
- *MSOL*-formulae $\psi_{1,1}, \dots, \psi_{1,j_1}, \dots, \psi_{\ell,1}, \dots, \psi_{\ell,j_\ell}$
- *MSOL*-formulae $\psi_{I,1}, \dots, \psi_{I,j_I}$

such that for every \mathcal{A}, \mathcal{I} and \mathcal{B}_i as above with $\mathcal{B}_i \models \psi_{i,j}$ iff $b_{i,j} = 1$ and $\mathcal{B}_I \models \psi_{I,j}$ iff $b_{I,j} = 1$ we have

$$\mathcal{A} \models \phi \text{ iff } F_\phi(b_{1,1}, \dots, b_{1,j_1}, \dots, b_{\ell,1}, \dots, b_{\ell,j_\ell}, b_{I,1}, \dots, b_{I,j_I}) = 1.$$

Moreover, F_ϕ and the $\psi_{i,j}$ are computable from ϕ , ℓ and vocabularies alone, but are tower exponential in the quantifier rank of ϕ .

Note that in most real applications, F_ϕ and the $\psi_{\alpha,j}$ are single exponential in the quantifier rank of ϕ .

Proof: The proof is classical, see in particular [28].

Now, we introduce an abstract preservation property of *XX*-combination of logics $\mathcal{L}_1, \mathcal{L}_2$, denoted by *XX* – *PP*($\mathcal{L}_1, \mathcal{L}_2$). *XX* may mean, for example, Disjoint Union. The property says roughly that if two *XX*-combinations of structures $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{B}_1, \mathcal{B}_2$ satisfy the same sentences of \mathcal{L}_1 then the disjoint unions $\mathcal{A}_1 \sqcup \mathcal{A}_2$ and $\mathcal{B}_1 \sqcup \mathcal{B}_2$ satisfy the same sentences of \mathcal{L}_2 . The reason we look at this abstract property is that the property *XX* – *PP*($\mathcal{L}_1, \mathcal{L}_2$) and its variants play an important role in our development of the Feferman-Vaught style theorems. This abstract approach was initiated by [23] and further developed in [29][30]. Now, we spell out various ways in which the theory of a disjoint union depends on the theory of the components. First, we look at the case, where the index structure is fixed.

Definition 4 (Preservation Property with Fixed Index Set):

For two logics \mathcal{L}_1 and \mathcal{L}_2 we define

Disjoint Pair

Input of operation: Two structures;

Preservation Property: if two pairs of structures $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{B}_1, \mathcal{B}_2$ satisfy the same sentences of \mathcal{L}_1 then the disjoint unions $\mathcal{A}_1 \sqcup \mathcal{A}_2$ and $\mathcal{B}_1 \sqcup \mathcal{B}_2$ satisfy the same sentences of \mathcal{L}_2 .

Notation: $P - PP(\mathcal{L}_1, \mathcal{L}_2)$

Disjoint Union

Input of operation: Indexed set of structures;

Preservation Property: if for each $i \in I$ (index set) \mathcal{A}_i and \mathcal{B}_i satisfy the same sentences of \mathcal{L}_1 then the disjoint unions $\bigsqcup_{i \in I} \mathcal{A}_i$ and $\bigsqcup_{i \in I} \mathcal{B}_i$ satisfy the same sentences of \mathcal{L}_2 .

Notation: $DJ - PP(\mathcal{L}_1, \mathcal{L}_2)$

The *Disjoint Union* of a family of structures is the simplest example of juxtaposing structures where none of the components are linked to each other. Another way of producing a new structure from several given structures is by mixing (shuffling) structures according to a (definable) prescribed way along the index structure.

Definition 5 (Shuffle over Partitioned Index Structure):

Let \mathcal{I} be a partitioned index structure into β parts, using unary predicates $I_\alpha, \alpha < \beta$. Let $\mathcal{A}_i, i \in I$ be a family of structures such that for each $i \in I_\alpha$ $\mathcal{A}_i \cong \mathcal{B}_\alpha$, according to the partition. In this case, we say that $\bigsqcup_{i \in I} \mathcal{A}_i$ is the *shuffle of \mathcal{B}_α along the partitioned index structure \mathcal{I}* , and denote it by $\bigsqcup_{\alpha < \beta}^{\mathcal{I}} \mathcal{B}_\alpha$.

Note that the shuffle operation, as defined here, is a special case of the disjoint union, and that the disjoint pair is a special case of the finite shuffle.

In the case of variable index structures and of *FOL*, Feferman and Vaught observed that it is not enough to look at the *FOL*-theory of the index structures, but one has to look at the *FOL*-theories of expansions of the Boolean algebras $PS(\mathcal{I})$ and $PS(\mathcal{J})$ respectively. *PS* is used for *Power Set*.

Gurevich suggested another approach, by looking at the *MSOL* theories of structures \mathcal{I} and \mathcal{J} . This is really the same, but more in the spirit of the problem, as the passage from \mathcal{I} to an expansion of $PS(\mathcal{I})$ remains on the semantic level, whereas the comparison of theories is syntactic. There is not much freedom in choosing the logic in which to compare the index structures, so we assume it always to be *MSOL*.

Definition 6 (PP with Variable Index Structures):

For two logics \mathcal{L}_1 and \mathcal{L}_2 we define

Disjoint Multiples

Input of operation: Structure and Index structure;

Preservation Property: Given two pairs of structures \mathcal{A}, \mathcal{B} and \mathcal{I}, \mathcal{J} such that \mathcal{A}, \mathcal{B} satisfy the same sentences of \mathcal{L}_1 and \mathcal{I}, \mathcal{J} satisfy the same *MSOL*-sentences. Then the disjoint unions $\bigsqcup_{i \in I} \mathcal{A}$ and $\bigsqcup_{j \in J} \mathcal{B}$ satisfy the same sentences of \mathcal{L}_2 .

Notation: $Mult - PP(\mathcal{L}_1, \mathcal{L}_2)$

Shuffles

Input of operation: A family of structures $\mathcal{B}_\alpha : \alpha < \beta$ and a (finitely) partitioned index structure \mathcal{I} with I_α a partition.

Preservation Property: Assume that for each $\alpha < \beta$ the pair of structures $\mathcal{A}_\alpha, \mathcal{B}_\alpha$ satisfy the same sentences of \mathcal{L}_1 , and \mathcal{I}, \mathcal{J} satisfy the same *MSOL*-sentences. Then the shuffles $\bigsqcup_{\alpha < \beta}^{\mathcal{I}} \mathcal{A}_\alpha$ and $\bigsqcup_{\alpha < \beta}^{\mathcal{J}} \mathcal{B}_\alpha$ satisfy the same sentences of \mathcal{L}_2 .

Notation: $Shu - PP(\mathcal{L}_1, \mathcal{L}_2)$ or for finite shuffles: $FShu - PP(\mathcal{L}_1, \mathcal{L}_2)$.

Observation 1:

Assume that for two logics $\mathcal{L}_1, \mathcal{L}_2$ we have the preservation property $XX - PP(\mathcal{L}_1, \mathcal{L}_2)$ and \mathcal{L}'_1 is an extension of \mathcal{L}_1 , \mathcal{L}'_2 is a sub-logic of \mathcal{L}_2 , then $XX - PP(\mathcal{L}'_1, \mathcal{L}'_2)$ holds as well.

Observation 2:

For two logics $\mathcal{L}_1, \mathcal{L}_2$ the following implications between preservation properties hold: $DJ - PP(\mathcal{L}_1, \mathcal{L}_2)$ implies $P - PP(\mathcal{L}_1, \mathcal{L}_2)$ and, for fixed index structures, $Mult - PP(\mathcal{L}_1, \mathcal{L}_2)$, $Shu - PP(\mathcal{L}_1, \mathcal{L}_2)$ and $FShu - PP(\mathcal{L}_1, \mathcal{L}_2)$. Moreover, for variable index structures we have $Shu - PP(\mathcal{L}_1, \mathcal{L}_2)$ implies $FShu - PP(\mathcal{L}_1, \mathcal{L}_2)$ and $Mult - PP(\mathcal{L}_1, \mathcal{L}_2)$.

Definition 7 (Reduction Sequence for Shuffling):

Let \mathcal{I} be a finitely partitioned τ_{ind} -index structure and \mathcal{L} be logic. Let $\mathcal{A} = \bigsqcup_{\alpha < \beta}^{\mathcal{I}} \mathcal{B}_\alpha$ be the τ -structure which is the finite shuffle of the τ_α -structures \mathcal{B}_α over \mathcal{I} . A \mathcal{L}_1 -reduction sequence for shuffling for $\phi \in \mathcal{L}_2(\tau_{shuffle})$ is given by

- 1) a boolean function $F_\phi(b_{1,1}, \dots, b_{1,j_1}, \dots, b_{I,j_I})$
- 2) set Υ of \mathcal{L}_1 -formulae $\Upsilon = \{\psi_{1,1}, \dots, \psi_{\beta,j_\beta}\}$
- 3) *MSOL*-formulae $\psi_{I,1}, \dots, \psi_{I,j_I}$

and has the property that for every \mathcal{A}, \mathcal{I} and \mathcal{B}_α as above with $\mathcal{B}_\alpha \models \psi_{\alpha,j}$ iff $b_{\alpha,j} = 1$ and $\mathcal{B}_I \models \psi_{I,j}$ iff $b_{I,j} = 1$ we have

$$\mathcal{A} \models \phi \text{ iff } F_\phi(b_{1,1}, \dots, b_{1,j_1}, \dots, b_{\beta,1}, b_{I,j_I}) = 1.$$

Note that we require that F_ϕ and the $\psi_{\alpha,j}$'s depend only on ϕ, β and $\tau_1, \dots, \tau_\beta$ but not on the structures involved.

Now, we list which Preservation Properties hold for which fuzzy logics.

Theorem 2:

Let \mathcal{I} be an index structure and \mathcal{L} be a fuzzy logic with either *lub* and *glb*, defined as set intersection and union [19], or *lub* corresponds to *max*, *glb* corresponds to *min* [20].

Then $DJ - PP(\mathcal{L}, \mathcal{L})$ and $FShu - PP(\mathcal{L}, \mathcal{L})$ hold.

Proof:

\cap, \cup : The proof by analyzing and extension of the proof in [23], [31].

max, min: The proof by analyzing and extension of the proof in [31], [32].

III. SYNTACTICALLY DEFINED TRANSLATION SCHEMES

The second logical reduction technique that we use is the *syntactically defined translation schemes*, which describe transformations of logical structures. The notion of abstract translation schemes comes back to Rabin [24]. They give rise to two induced maps, translations and transductions. Transductions describe the induced transformation of logical structures and the translations describe the induced transformations of logical formulae.

Definition 8 (Translation Schemes Φ):

Let τ_1 and τ_2 be two vocabularies and \mathcal{L} be a logic. Let $\tau_2 = \{R_1, \dots, R_m\}$ and let $\rho(R_i)$ be the arity of R_i . Let $\Phi = \langle \varphi, \psi_1, \dots, \psi_m \rangle$ be formulae of $\mathcal{L}(\tau_1)$. Φ is κ -feasible for τ_2 over τ_1 if φ has exactly κ distinct free variables and each ψ_i has $\kappa \rho(R_i)$ distinct free variables. Such a $\Phi = \langle \varphi, \psi_1, \dots, \psi_m \rangle$ is also called a κ - τ_1 - τ_2 -translation scheme

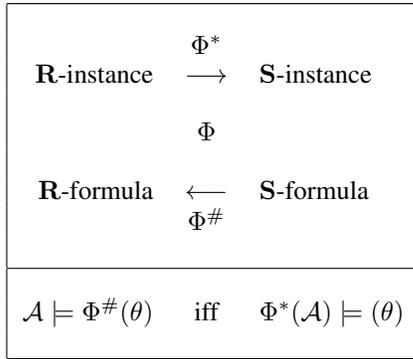


Figure 1. Components of translation schemes and the fundamental property

or, shortly, a *translation scheme*, if the parameters are clear in the context.

In general, Definition 8 must be adopted to the given fuzzy logic \mathcal{L} , if it is not straightforward. For a fuzzy logic \mathcal{L} with a translation scheme Φ we can *naturally* associate a (partial) function Φ^* from τ_1 -structures to τ_2 -structures.

Definition 9 (The induced map Φ^):*

Let \mathcal{A} be a τ_1 -structure with universe A and Φ be κ -feasible for τ_2 over τ_1 . The structure \mathcal{A}_Φ is defined as follows:

- 1) The universe of \mathcal{A}_Φ is the set $A_\Phi = \{\bar{a} \in A^\kappa : \mathcal{A} \models \varphi(\bar{a})\}$.
- 2) The interpretation of R_i in \mathcal{A}_Φ is the set

$$A_\Phi(R_i) = \{\bar{a} \in A_\Phi^{\rho(R_i) \cdot \kappa} : \mathcal{A} \models \psi_i(\bar{a})\}.$$

Note that \mathcal{A}_Φ is a τ_2 -structure of cardinality at most $|A|^\kappa$.

- 3) The partial function $\Phi^* : Str(\tau_1) \rightarrow Str(\tau_2)$ is defined by $\Phi^*(\mathcal{A}) = \mathcal{A}_\Phi$. Note that $\Phi^*(\mathcal{A})$ is defined iff $\mathcal{A} \models \exists \bar{x} \varphi$.

For fuzzy logic \mathcal{L} with a translation scheme Φ we can also *naturally* associate a function $\Phi^\#$ from $\mathcal{L}(\tau_2)$ -formulae to $\mathcal{L}(\tau_1)$ -formulae.

Definition 10 (The induced map $\Phi^\#$):

Let θ be a τ_2 -formula and Φ be κ -feasible for τ_2 over τ_1 . The formula θ_Φ is defined inductively as follows:

- 1) For $R_i \in \tau_2$ and $\theta = R(x_1, \dots, x_m)$ let $x_{j,h}$ be new variables with $i \leq m$ and $h \leq \kappa$ and denote by $\bar{x}_i = \langle x_{i,1}, \dots, x_{i,\kappa} \rangle$. We put $\theta_\Phi = \psi_i(\bar{x}_1, \dots, \bar{x}_m)$.
- 2) For the boolean connectives the translation distributes, i.e. if $\theta = (\theta_1 \vee \theta_2)$ then $\theta_\Phi = (\theta_{1\Phi} \vee \theta_{2\Phi})$ and if $\theta = \neg \theta_1$ then $\theta_\Phi = \neg \theta_{1\Phi}$, and similarly for \wedge .
- 3) For the existential quantifier, we use relativization, i.e. if $\theta = \exists y \theta_1$, let $\bar{y} = \langle y_1, \dots, y_\kappa \rangle$ be new variables. We put $\theta_\Phi = \exists \bar{y} (\varphi(\bar{y}) \wedge \theta_{1\Phi})$.
- 4) For (monadic) second order variables U of arity ℓ ($\ell = 1$ for *MSOL*) and \bar{v} a vector of length ℓ of first order variables or constants we translate $U(\bar{v})$ by treating U like a relation symbol above and put $\theta_\Phi = \exists V (\forall \bar{v} (V(\bar{v}) \rightarrow (\phi(\bar{v}_1) \wedge \dots \wedge \phi(\bar{v}_\ell) \wedge (\theta_1)_\Phi)))$.
- 5) The function $\Phi^\# : \mathcal{L}(\tau_2) \rightarrow \mathcal{L}(\tau_1)$ is defined by $\Phi^\#(\theta) = \theta_\Phi$.

Observation 1: If we use *MSOL* and Φ^* is over *MSOL* too, and it is vectorized, then we do not obtain *MSOL* for \mathcal{A}_Φ . In most of feasible applications, we have that Φ^* is not vectorized, but not necessarily.

Observation 2:

- 1) $\Phi^\#(\theta) \in$ fuzzy *FOL* (*FFOL*) provided $\theta \in$ *FFOL*, even for vectorized Φ .
- 2) $\Phi^\#(\theta) \in$ *MSOL* provided $\theta \in$ *MSOL*, but only for scalar (non-vectorized) Φ .

The following fundamental theorem is easily verified for correctly defined \mathcal{L} translation schemes, see Figure 1. Its origins go back at least to the early years of modern logic [33, page 277 ff]. See also [21].

Theorem 3:

Let $\Phi = \langle \varphi, \psi_1, \dots, \psi_m \rangle$ be a κ - τ_1 - τ_2 -translation scheme, \mathcal{A} a τ_1 -structure and θ a $\mathcal{L}(\tau_2)$ -formula. Then

$$\mathcal{A} \models \Phi^\#(\theta) \text{ iff } \Phi^*(\mathcal{A}) \models \theta.$$

IV. STRONGLY DISTRIBUTED FUZZY STRUCTURES

The disjoint union and shuffles as such are not very interesting. However, combining them with translation schemes gives as a rich repertoire of composition techniques. Now, we generalize the disjoint union or shuffling of fuzzy structures to *Strongly Distributed Fuzzy Structures* in the following way:

Definition 11 (Strongly Distributed Fuzzy Structures):

Let \mathcal{I} be a finitely partitioned index structure and \mathcal{L} be *FFOL*. Let $\mathcal{A} = \bigsqcup_{i \in \mathcal{I}} \mathcal{A}_i$ be a τ -structure, where each \mathcal{A}_i is isomorphic to some $\mathcal{B}_1, \dots, \mathcal{B}_\beta$ over the vocabularies $\tau_1, \dots, \tau_\beta$, in accordance with the partition. For a Φ be a τ_1 - τ_2 \mathcal{L} -translation scheme, the Φ -*Strongly Distributed Fuzzy Structure*, composed from $\mathcal{B}_1, \dots, \mathcal{B}_\beta$ over \mathcal{I} is the structure $\Phi^*(\mathcal{A})$, or rather any structure isomorphic to it.

Now, our main Theorem 4 can be formulated as follows:

Theorem 4:

Let \mathcal{I} be a finitely partitioned index structure, \mathcal{L} be *FFOL* such that Theorem 2 holds for it. Let \mathcal{S} be a Φ -*Strongly Distributed Fuzzy Structure*, composed from $\mathcal{B}_1, \dots, \mathcal{B}_\beta$ over \mathcal{I} , as above. For every $\phi \in \mathcal{L}(\tau)$ there are

- 1) a boolean function $F_{\Phi, \phi}(b_{1,1}, \dots, b_{1,j_1}, \dots, b_{I,j_I})$,
- 2) \mathcal{L} -formulae $\psi_{1,1}, \dots, \psi_{1,j_1}, \dots, \psi_{\beta,1}, \dots, \psi_{\beta,j_\beta}$ and
- 3) *MSOL*-formulae $\psi_{I,1}, \dots, \psi_{I,j_I}$

such that for every \mathcal{S}, \mathcal{I} and \mathcal{B}_i as above with $\mathcal{B}_i \models \psi_{i,j}$ iff $b_{i,j} = 1$ and $\mathcal{I} \models \psi_{I,j}$ iff $b_{I,j} = 1$ we have

$$\mathcal{S} \models \phi \text{ iff } F_{\Phi, \phi}(b_{1,1}, \dots, b_{1,j_1}, \dots, b_{I,j_I}) = 1.$$

Moreover, $F_{\Phi, \phi}$ and $\psi_{i,j}$ are computable from $\Phi^\#$ and ϕ , but are tower exponential in the quantifier rank of ϕ .

Proof: By analyzing the proof of Theorem 2 with Theorem 3.

Now, we provide an example of the applicability of our approach. Let us consider the following composition of two input graphs H and G . G can be viewed as a display graph, where on each node we want to have a copy of H , such that certain additional edges between the copies are added. In practice, this is an extended model on message passing. The nodes marked with L^j are the communication ports.

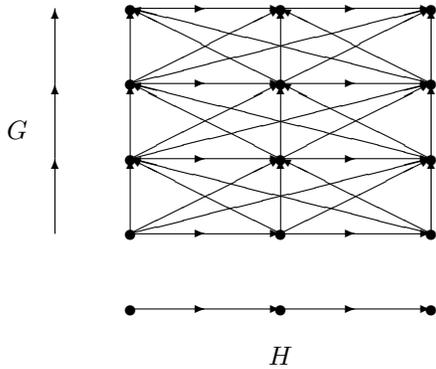


Figure 2. Uniform graph substitution.

Let $G = \langle V_G, R \rangle$ and $H = \langle V_H, S, L^j (j \in J) \rangle$ be two relational structures (J is finite), then their composition

$$S = \langle V_S, L_S^1, \dots, L_S^{|J|}, S_S, R_S^j (j \in J) \rangle$$

is defined as follows, see Figure 2:

- $V_S = \dot{\bigcup}_{g \in G} V_H^g$, where each V_H^g is isomorphic to V_H ;
- $L_S^j(w)$ is true if w belongs to L^j ;
- $S_S = \{(w, v) : w \in V_H^g, v \in V_H^g, S(w, v)\}$;
- $R_S^j = \{(w, v) : L^j(w), L^j(v), P(i, w), P(i', v), R(i, i')\}$.

It is easy to see (Figure 2) that this construction can be obtained from the Cartesian product $G \times H$ by a *FOL* translation scheme without vectorization. However, the Cartesian product cannot be obtained from $\dot{\bigcup}_{g \in G} H$ without vectorization. S can be obtained from the disjoint union $\dot{\bigcup}_{g \in G} H$ by a *FOL* translation scheme. The following proposition makes it precise.

Proposition 1:

S is isomorphic to $\Phi^*(\dot{\bigcup}_{g \in G} H)$ with

$$\Phi = \langle \phi, \psi_{L_S^1}, \dots, \psi_{L_S^{|J|}}, \psi_S, \psi_{R_S^1}, \dots, \psi_{R_S^{|J|}} \rangle, \text{ and}$$

$$\begin{aligned} \phi &= \exists i(P(i, x) \wedge \text{Index}(i)), \\ \psi_{L_S^j} &= \exists i((P(i, x) \wedge \text{Index}(i)) \wedge L^j(x)), \\ \psi_S &= \exists i((\text{Index}(i) \wedge (P(i, w) \wedge P(i, v))) \wedge S(w, v)), \\ \psi_{R_S^j} &= \exists i \exists i'(((\text{Index}(i) \wedge \text{Index}(i')) \wedge R(i, i')) \wedge (L^j(w) \wedge L^j(v))) \wedge (P(i, w) \wedge P(i', v))). \end{aligned}$$

In this example, depending on the choice of the interpretation of the L^j 's, more sophisticated parallel data communication systems can be modeled, but not all.

V. OUTLINE OF THE METHOD

Our general scenario is as follows: given a strongly distributed fuzzy structure \mathcal{A} with index structure \mathcal{I} . Formula ϕ of fuzzy logic must be evaluated on \mathcal{A} .

The question is: *What is the reduction sequence of ϕ if any?*

Here, we propose a general approach to answer the question and to compute the reduction sequences algorithmically. The general template is defined as follows:

A. Prove preservation theorems

Given fuzzy logic \mathcal{L} .

1) *Define disjoint union of \mathcal{L} -structures:* In the general case, we use Definition 2 of *Disjoint Union (DJ)* of the components: $\mathcal{A} = \dot{\bigcup}_{i \in I} \mathcal{A}_i$.

2) *Define a preservation property $XX - PP$ for \mathcal{L} :* After we introduced the appropriate disjoint union of structures, we define the notion of a *Preservation Property (PP)* for the fuzzy logic.

3) *Prove the $XX - PP$ for \mathcal{L} :* Results like Theorem 2 are not always true as it was shown in [26].

B. Define Translation Schemes

Given fuzzy logic \mathcal{L} . Definition 8 introduces the classical *syntactically defined translation schemes* [24]. Definitions 8 gives rise to two induced maps: translations and transductions. Transduction Φ^* describes the induced transformation of \mathcal{L} -structures and the translation $\Phi^\#$ describes the induced transformations of logical formulae. The fundamental Theorem 3 should hold for correctly defined \mathcal{L} translation schemes.

C. Strongly Distributed Fuzzy Structures

Given a \mathcal{L} -structure \mathcal{A} . At this step, we have defined disjoint unions (and shuffles) of \mathcal{L} -structures. Using translation scheme Φ , we introduce the notion of *Strongly Distributed Fuzzy Structures* in Definition 11. Now, the proof of theorems like Theorem 4 should pretty straightforward and provides the desired reduction sequence. In fact, $F_{\Phi, \phi}$ and the $\psi_{i,j}$ of Theorem 4 are computable from $\Phi^\#$ and ϕ . However, we note that they are tower exponential in the quantifier rank of ϕ .

D. Incremental Reasoning on Strongly Distributed Fuzzy Systems

Finally, we derive a method for evaluating \mathcal{L} -formula ϕ on \mathcal{A} , which is a Φ -strongly distributed fuzzy composition of its components. The method proceeds as follows:

Preprocessing: Given ϕ and Φ , but not a \mathcal{A} , we algorithmically construct a sequence of formulae $\psi_{i,j}$ and an evaluation function $F_{\Phi, \phi}$ as in Theorem 4.

Incremental Computation: We compute the local values $b_{i,j}$ for each component of the \mathcal{A} .

Final Solution: Now, Theorem 4 states that ϕ , expressible in the corresponding fuzzy logic \mathcal{L} , on \mathcal{A} may be effectively computed from $b_{i,j}$, using $F_{\Phi, \phi}$.

VI. CONCLUSION AND OUTLOOK

In this work, we introduced the notion of strongly distributed fuzzy systems and presented a uniform approach to incremental automated reasoning on such systems. The approach is based on systematic use of two logical reduction techniques: Feferman-Vaught reductions and the syntactically defined translation schemes.

Our general scenario is as follows: Given a fuzzy structure \mathcal{A} that is composed from structures \mathcal{A}_i ($i \in I$) and index structure \mathcal{I} . A formula ϕ of fuzzy logic \mathcal{L} describes a property to be checked on \mathcal{A} . The question is: What is the reduction sequence for ϕ , if any such a sequence exists?

We showed that if we can prove preservation theorems for \mathcal{L} as well as if \mathcal{A} is a strongly distributed composition of its

components, then the corresponding reduction sequence for \mathcal{A} can be effectively computed algorithmically. In such a case, we derive a method for evaluating an \mathcal{L} -formula ϕ on \mathcal{A} , which is a Φ -strongly distributed composition of its components.

First, given ϕ and Φ , but not a \mathcal{A} , we algorithmically construct a sequence of formulae $\psi_{i,j}$ and an evaluation function $F_{\Phi,\phi}$. Next, we compute the local values $b_{i,j}$ for each component of the \mathcal{A} . Finally, our main theorems state that ϕ , expressible in the corresponding logic \mathcal{L} on \mathcal{A} , is effectively computed from $b_{i,j}$, using $F_{\Phi,\phi}$.

We plan to apply the proposed methodology to the incremental reasoning, based on the promising variations of *WMSOL* as introduced recently in [34] [35] [36] (see also [37]).

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