# Testing the representative capacity of parties and coalitions (with applications to German Bundestag)

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***Abstract*—Five leading German parties and their coalitions are evaluated from the viewpoint of direct democracy. For this purpose, the positions of the parties on over 30 topical issues, as given for the last Bundestag (parliament) elections 2009, are compared with the outcomes of public opinion polls. The results are summarized in the party indices of universality (percentage of issues with majority representation). The same is done for party coalitions. A statistical test is developed to judge whether the index magnitudes are sufficiently high to confirm the representative capacity. It is shown that the representativeness of German parties and their coalitions is statistically insignificant.**

***Keywords*-Mathematical theory of democracy; statistical test; parties; coalitions; representativeness; Bernoulli matrices; sums of random vectors.**

## I. Introduction

The mathematical theory of democracy provides methods to evaluate single representatives (candidates for president, political parties) and representative bodies (parliament, cabinet of ministers) regarding their capacity to express opinions of the population. The evaluation is based on comparing the position of representatives on selected policy issues with the public opinion revealed in public polls, referenda, or plebiscites. However, any conclusion based on a data sample has a limited reliability. Therefore, the statistical significance of evaluation has to be estimated.

To illustrate what we are going to study, suppose that five political parties define their position on six issues like ‘Introduce nation-wide minimum wage’, ‘Privatize railways’, etc., and, according to public opinion polls, one party perfectly represents the public opinion, matching the majority opinion on all the issues. The party looks highly representative, nevertheless, the following questions emerge:

1) Does the outcome observed really indicate at the party’s representative capacity, or it may be just a coincidence by chance? In other words, can a similar performance be expected on other policy issues, not yet considered or arising in future?
2) Are six policy issues sufficient to make any conclusion about the party’s representative capacity? Or their number should be increased, say, to 10?
3) What can be said if the match of party positions to the public opinion is imperfect, for instance, is restricted to five out of six issues? Does the conclusion about the party’s representativeness remain valid?

The same questions can be addressed to party coalitions.

The usual approach to this type of problems is developing a statistical test. Assuming that the parties meet the public opinion randomly, the probability of the actual outcome is found. If it is small then a coincidence by chance looks improbable and the actual observation is interpreted as a manifestation of the party’s representative capacity. If the probability is not small then the outcome looks possible and no conclusion on the party capacity is made.

The match of party positions to public opinion can be represented by a table, in our example of six issues versus five parties, with 1s standing for match and 0s for no match. If the match is assumed random, the table elements turn into independent Bernoulli random variables, taking values 0 and 1 with equal probabilities 1/2.

If an alone-standing party is considered then the table consists of a single column. The match on six out of six issues has the probability $(1/2)^6 = 1/64 \approx 0.02$. Here, 0.02 is the significance level of the null hypothesis — that there is no representative capacity, i.e., the match and non-match of the party positions with the public opinion is unpredictable, and the responses to the questions are independent. In social sciences it is traditional to use the 5% significance threshold; therefore, 0.02 is considered too small, the null hypothesis is rejected and the alternative hypothesis about the existing representative capacity is accepted. Hence, the party is regarded representative.

For five parties, the situation is different. The probability that one party out of five expresses the majority opinion on all the six issues is $1 - \left[1 - (1/2)^6\right]^5 \approx 0.08$. It is not small enough to say that the actual outcome is little probable, so the party’s representative capacity is in question. However, if the perfect match is observed for seven out of seven issues then the probability $1 - \left[1 - (1/2)^7\right]^5 \approx 0.04$ is sufficiently small, arguing for the party’s representativeness.

Making conclusions about coalitions is similar, but random coincidences are more frequent than for single parties.

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In our example of five parties, the occurrence of a three-party coalition which represents the public opinion on six out of six issues has the probability of about $0.10$. Hence, the perfect coalition performance observed is not much promising for the future. The common probability threshold 0.05 (= 5%-significance level) can be surpassed with as many as eight hits out of eight. If the match is imperfect then the sample of issues should be extended further. For three-party coalitions, a single mismatch on $i = 1$ issue must be outbalanced by at least $m - i = 10$ hits, otherwise the 5%-significance is not attained.

Computing the probabilities required to statistically 'prove' the representative capacity of coalitions is not easy, and just this task is the subject of this paper.

## II. Problem formulation

*Perfect column pairs and column triplets:* A *Bernoulli* $(m \times n)$*-matrix* $B = \{b_{ij}\}$ is a matrix whose elements $b_{ij}$ are independent Bernoulli random variables, taking values 0 and 1 with equal probabilities 1/2. A $k$-tuple of its columns is called *perfect* if its sum along rows is a column $m$-vector with all $m$ elements being $\geq k/2$ .

Label every $k$-tuple of columns of Bernoulli matrix with the set of corresponding column numbers $J = [j_1, \ldots, j_k]$. Order these labels $J$ and use them as scalar indices of column $k$-tuples.

By $A_J$ denote the event that the $J$-th $k$-tuple is perfect. We are interested in the probability of union of these events, meaning that there occurs at least one perfect $k$-tuple of columns:

$$\Pr\left(\bigcup A_J\right) = ? \tag{1}$$

A table with random 0–1 codes of match of party positions to public opinion is nothing else but a Bernoulli matrix. Here, $m$ rows are associated with $m$ issues, and $n$ columns are associated with $n$ parties. If the majority opinion on the $i$-th issue is represented by the $j$-th party then the matrix element $b_{ij} = 1$, otherwise $b_{ij} = 0$.

A perfect $k$-tuple of columns corresponds to a coalition of $k$ parties whose internal majority ($\geq k/2$ parties) shares the prevailing public opinion on every issue. The probability (1) characterizes the occurrence of such coalitions by chance and is needed to statistically test the representative capacity of coalitions with 100%-representativeness observed. That is, it is addressed to answer Questions 1–2.

***i**-imperfect column pairs and column triplets:* To study Question 3 about imperfect match of party positions to public opinion, weaken the perfectness-condition. If it is violated in $i$ or fewer rows, the $k$-tuple of columns is called $i$*-imperfect*, that is, its sum along rows is a column $m$-vector with at least $m - i$ elements being $\geq k/2$ .

Obviously, perfect $k$-tuples of columns are $0$-imperfect. A $i$-imperfect $k$-tuple of columns corresponds to a coalition which represents the majority opinion incompletely, failing to do it on $i$ or fewer issues. The events $A_J$ and the probability (1) are respectively redefined for $i$-imperfect $k$-tuples of columns.

*Existing literature:* Besides the mathematical theory of democracy, the problem of estimating the probabilities mentioned arises in genetics, logistics, and some other applications like traffic control or finances [5], [6], [13], [14], [15]. Random matrices are considered in numerous publications; for a survey see [7], [8], [9], [10]. In particular, there are papers focused on sums of random vectors and their approximations; see [1], [2], [3], [5], [11].

These publications study trends in large random matrices or in large sums of random vectors rather than propose solutions for small and medium-sized practical applications where asymptotic properties are not salient. The given paper attempts to fill in this gap by developing approaches to the problem for column pairs and column triplets in small and medium-sized Bernoulli matrices, that is, for coalitions with two or three parties if the total number of parties and the number of reference policy issues are rather limited.

*Meta-modeling approach:* For Bernoulli matrices, three ways to find the probability (1) are developed. One method is geometric, another algebraic, and the third properly probabilistic. In theory, each of these methods solves the problem, but in practice every method has its computational limits. The geometric solution is computationally appropriate for Bernoulli matrices with a few columns, the algebraic — for Bernoulli matrices with a few rows, and the probabilistic — for Bernoulli matrices with twice more rows than columns. Therefore, the united computational solution is combined from the three methods. There are still non-computable probabilities, and their approximations are estimated from the known probabilities by five interpolation techniques.

The general approach is based on meta-modelling. Each meta-model builds a series of models with computational formulas for particular sizes of the Bernoulli matrix. These formulas are too complex to be derived 'manually' and have no visible regularity, so the meta-modeling approach is essential.

The complexity and lack of regularities may evoke suspects in the model errors. The doubts are resolved by equal output from different methods. In fact, the probabilities computed by alternative methods, say, geometric and algebraic, coincide with the precision better than $\epsilon = 2^{-25}$.

*About this paper:* This paper focuses on the application of the statistical test to estimate the statistical significance of the representativeness of five German parties currently in the Bundestag and their coalitions. The full account of mathematical methods which back up the statistical test is presented in [16]. A more broad overview of the mathematical theory of democracy is given in [14], [17], [18].

## III. Application to German Parties

Let us come back to the questions posed in Introduction. Apply the results obtained to evaluate the representative capacity of five eligible German parties and their possible coalitions at the time of Bundestag elections 2009; see Table I.

Figure 1 shows positions of the five parties on 32 topical policy issues; as well as outcomes of polls of public opinion on these issues. The party positions are taken from the Wahl-O-Mat — a German internet site developed after a similar Dutch site *StemWijzer* (Vote match) of the late 1990s [12], [19]. These sites were designed to stimulate political participation, primarily by young people. The user fills in a questionnaire on topical political issues with Yes/No answers, eventually with weights; then the answers are compared with the answers of the parties, and the user learns which party fits best to his political profile [4]. Before the elections, the governmental supervising committee — Bundeszentrale für politische Bildung — officially receives from the parties their Yes/No answers to the questions for the Wahl-O-Mat. Therefore, the information about party positions we refer to is *official*. The answers of the Wahl-O-Mat users are unavailable, because they are not saved even as cumulated statistics. The position of the electorate on the issues is taken from related public opinion polls; see [16], [17] for references to data sources.

To explain the figure, consider the top question: ‘2. Introduce nation-wide minimal wage’. The question number ‘2’ is as in the ‘official’ *Wahl-O-Mat* table filled by the parties shortly before the Bundestag elections 2009. Each party is depicted by a rectangle, whose length is proportional to the number of the party seats in the Bundestag. The ‘No/Yes’ party opinion on the question is reflected by the location of the rectangle to the left side or to the right side from the central vertical axis, respectively. A Bundestag majority is attained if the cumulative length of party rectangles surpasses the 50%-threshold (marked with dotted lines). The balance of public opinion on each issue is shown by the blue bars with the length normalized to 100% (abstaining respondents are ignored). Their bias from the center indicates at the prevailing public opinion.

For every question, a given party represents either a majority, or a minority of the population (identified with the fraction in the opinion polls). For instance, the CDU-CSU (black rectangle) with the ‘No’ answer to the top question ‘2. Introduce nation-wide minimal wage’ represents the opinion of 43% of the population against 52%. After normalization, we obtain that its *representativeness* for question 2 is

$$r_{\text{CDU-CSU,2}} = \frac{43}{43+52} \cdot 100\% \approx 45\% \ .$$

Similarly, with the ‘No’ answer to the next question ‘17. Relax protection against dismissals’, the CDU-CSU expresses the opinion of 82% of the population against

Table I
Results of 2009 German parliamentary elections

| | CDU-CSU | SPD | FDP | Left-Party | Greens | 22 minor parties with $< 5\%$ of the votes |
|---|---|---|---|---|---|---|
| Percentage of votes | 33.8 | 23.0 | 14.6 | 11.9 | 10.7 | 6.0 |
| Bundestag seats, % | 36.0 | 24.5 | 15.5 | 12.7 | 11.4 | None |

CDU-CSU Christian Democratic Union together with Bavaria’s Christian Social Union (conservatives)
SPD Social Democratic Party
FDP Free Democratic Party (neoliberals) close to employer organizations
Left-Party fusion of the PDS (Party of Democratic Socialism—former East German communists) with the WASG (Voting Alternative for Employment and Social Justice—the separated left wing of the SPD)
Greens party of ecologists in a broad sense with a social-democratic background

17%. After normalization we obtain its representativeness for question 17

$$r_{\text{CDU-CSU,17}} = \frac{82}{82+17} \cdot 100\% \approx 83\% \ ,$$

and so on.

The frequency of representing a majority ($\geq 50\%$) is defined to be the *universality* of the party. As one can see, the CDU-CSU represents a majority on 15 questions from 32, having the degree of imperfectness $32 - 17 = 15$, or in %

$$\mathsf{U}_{\text{CDU-CSU}} = \frac{15}{32} \cdot 100\% \approx 47\% \ .$$

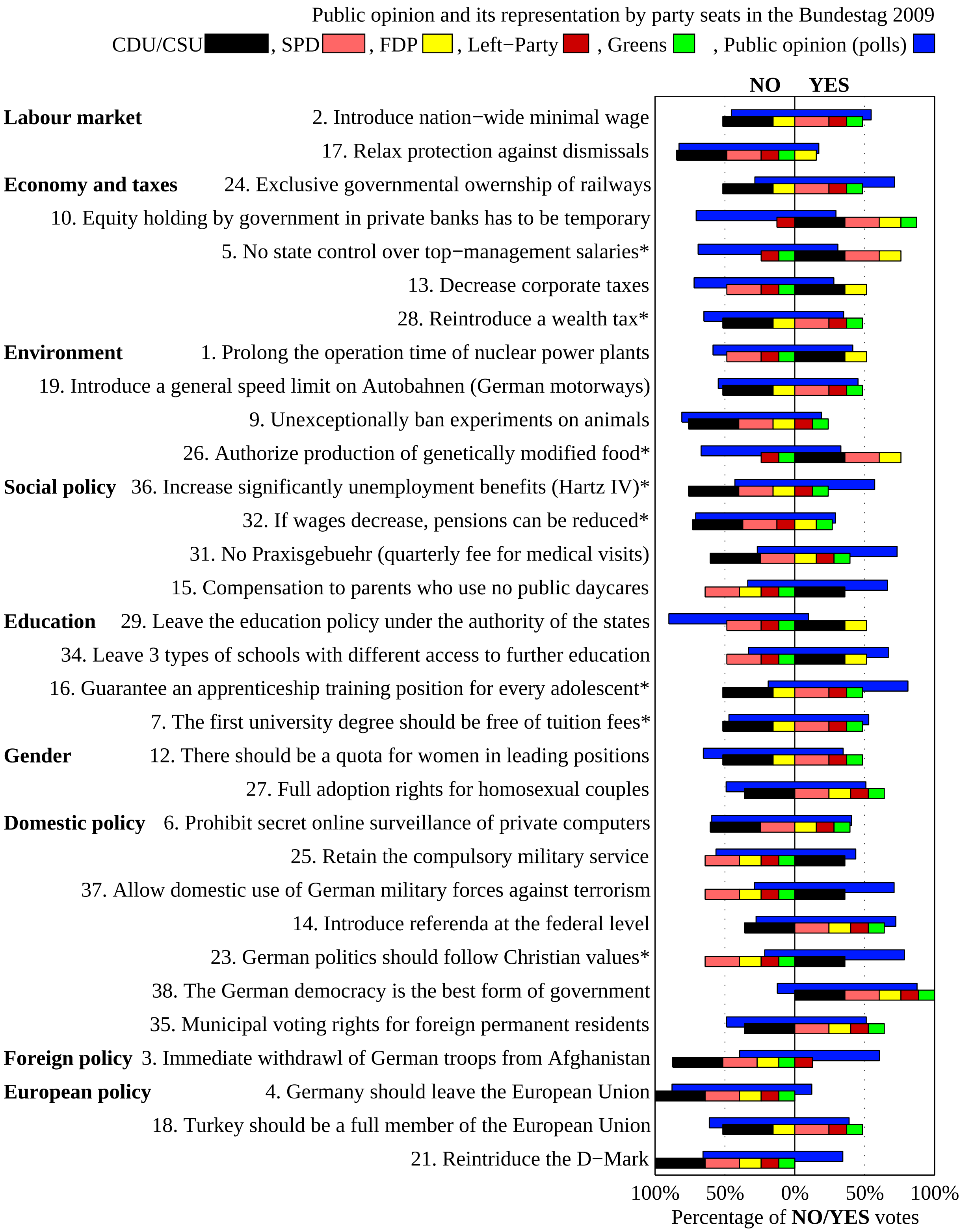
A higher universality means that a majority is represented more frequently. For instance the Left-Party represents a majority on 22 of 32 questions, having the degree of imperfectness $32 - 22 = 10$, or

$$\mathsf{U}_{\text{Left-Party}} = \frac{22}{32} \cdot 100\% \approx 69\% \ .$$

The universality indices and the degree of imperfectness of the parties are shown in Table II.

The representativeness indices for a coalition are computed with a probabilistic model. If the coalition member parties are unanimous on an issue, the coalition position on the issue is as that of the member parties. If the member parties are not unanimous, the index is computed for a lottery of two possible decisions on the issue, with the probability between 0.5 (absolute uncertainty) and the one which is proportional to the size of opposing factions within the coalition.

For the statistical test, however, a simplistic assumption is made: the position of a coalition on a given issue is defined by a simple majority of the member parties, for instance, if two parties out of three share the same opinion, this opinion is adopted by the coalition. In the case of a




* Adjustments based on party public statements, parliamentary voting, etc.

Figure 1. Public opinion and party positions on 32 issues

Table II
UNIVERSALITY INDICES OF PARTIES AND COALITIONS AND STATISTICAL SIGNIFICANCE OF THEIR REPRESENTATIVE CAPACITY. INTERPOLATED SIGNIFICANCE VALUES ARE **boldfaced**

| Single party or coalition | Universality, in % | Ranks | Degree of $i$-imperfectness | Significance | Adjusted seats |
|---|---|---|---|---|---|
| CDU-CSU | 46.9 | 13 | 17 | .9976 | 17.0 |
| SPD | 56.3 | 10 | 14 | .8299 | 20.5 |
| FDP | 43.8 | 14 | 18 | .9998 | 15.9 |
| Left-Party | 68.8 | 7 | 10 | .1191 | 25.0 |
| Greens | 59.4 | 9 | 13 | .6482 | 21.6 |
| CDU-CSU/SPD | 81.3 | 4 | 6 | **.7293** | 37.5 |
| CDU-CSU/FDP | 62.5 | 8 | 12 | **.9970** | 33.0 |
| CDU-CSU/Left-Party | 100.0 | 1 | 0 | .0010 | 42.0 |
| CDU-CSU/Greens | 93.8 | 2 | 2 | .0587 | 38.6 |
| SPD/FDP | 75.0 | 6 | 8 | **.9228** | 36.4 |
| SPD/Left-Party | 75.0 | 6 | 8 | **.9228** | 45.5 |
| SPD/Greens | 68.8 | 7 | 10 | **.9816** | 42.0 |
| FDP/Left-Party | 87.5 | 3 | 4 | .4072 | 40.9 |
| FDP/Greens | 78.1 | 5 | 7 | **.8534** | 37.5 |
| Left-Party/Greens | 68.8 | 7 | 10 | **.9816** | 46.6 |
| CDU-CSU/SPD/FDP | 50.0 | 12 | 16 | **.8505** | 53.4 |
| CDU-CSU/SPD/Left-Party | 56.3 | 10 | 14 | **.6298** | 62.5 |
| CDU-CSU/SPD/Greens | 56.3 | 10 | 14 | **.6298** | 59.1 |
| CDU-CSU/FDP/Left-Party | 50.0 | 12 | 16 | **.8505** | 58.0 |
| CDU-CSU/FDP/Greens | 46.9 | 13 | 17 | **.9139** | 54.5 |
| CDU-CSU/Left-Party/Greens | 62.5 | 8 | 12 | **.3327** | 63.6 |
| SPD/FDP/Left-Party | 56.3 | 10 | 14 | **.6298** | 61-4 |
| SPD/FDP/Greens | 53.1 | 11 | 15 | **.7561** | 58.0 |
| SPD/Left-Party/Greens | 62.5 | 8 | 12 | **.3327** | 67.0 |
| FDP/Left-Party/Greens | 59.4 | 9 | 13 | **.4820** | 62.5 |
| Bundestag 2009 | 50.0 | 13 | 16 | .5000 | 100.0 |
| Bundestag 2009 with adjusted seats | 56.3 | 10 | 14 | .2983 | 100.0 |

tie opinion in a two-party coalition, the prevailing public opinion is assumed decisive, as if influencing the internal coalition debate. After the positions of coalitions have been determined, their universality and imperfectness indices are defined in the same way as for single parties. These indices for possible two- and three-party coalitions are also shown in Table II.

The statistical significance of the representative capacity of single parties and properly coalitions is shown in the last column of Table II. Here, the representative capacity of no party and of no three-party coalition is even 10%-statistically significant, to say nothing of the usual 5%-threshold. As for two-party coalitions, the only 5%-significant representative capacity is inherent in the 'politically impossible' coalition of the CDU-CSU with its extreme political opponent, the Left-Party. Its 100-% universality results from our assumption that in the case of tie vote the coalition position is determined by the public opinion. On four issues the CDU-CSU and the Left-Party share the prevailing public opinion. On the remaining 28 issues they are opposite, which, according to our assumption, makes the coalition's opinion the prevailing public opinion again. Thereby the coalition 'perfectly' expresses the public opinion, which is of course a strained conclusion.

Thus, the representative capacity of German parties is statistically insignificant. A relatively high degree of imperfectness of match of their positions to the prevailing public opinion leaves little hope that new surveys with additional policy issues can change this conclusion. The same holds for the party coalitions; however, with reservations caused by a simplistic assumption of the statistical test that the coalition member parties adopt the position on an issue by the majority rule.

## IV. DISCUSSION: AN ALTERNATIVE ELECTION METHOD

In representative democracy, political participation by the people is realized through election of representatives. Therefore, representative democracy is *democratic* to the degree with which the elected *represent* the public interest.

To increase in the representativeness of a parliament, an election procedure with an alternative architecture is imagined. Electoral ballots are proposed to include questions about the voter's position on key issues in candidate manifestos (Introduce nationwide minimum wage? Yes/No, Relax protection against employee dismissals? Yes/No, etc.). The election method envisages processing the totality of the ballots and evaluating candidates by the degree to which their profiles match with that of the electorate as a single body. It differs from common elections in that candidates receive no votes. In contrast to voting based on individual choices, this procedure implements public determination. The embedded referenda on a sample of issues serve as a 'direct democracy test' of the candidates. In a sense, our proposal attempts to bridge direct and representative democracies, and to make election better meet democratic objectives.

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Let us illustrate our proposal with redistributing the seats in the German Bundestag, referring the German public profile based on 32 polls of public opinion on 32 policy issues and the political profiles of the five leading German parties shown in Figure 1. The degree of match of the parties with the public profile is expressed by the universality indicator given in Table II.

Now we make the size of the Bundestag factions proportionally to the party universality indices. For instance,

$$\text{Adjusted seats of CDU-CSU} = \frac{\overbrace{47}^{\substack{\text{Universality}\\ \text{of CDU-CSU}}} \times 100\%}{\underbrace{47+56+44+69+59}_{\substack{\text{Sum of universality indices}\\ \text{of the five leading parties}}}} = 17\% \ .$$

The adjusted seats for the five parties are shown in the last column of Table II. Note the increase in the Bundestag universality displlayed in the bottom row of the table.

Of course, this is only an illustration. The procedure can be modified arbitrarily and/or used in weighted combinations with traditional voting schemata. More generally, one can consider an optimization model to maximize the representativeness of the Bundestag (its universality index) by varying the size of Bundestag factions.

## V. Conclusions

Thus, the representativeness of candidates (parties) can be accurately tested from the viewpoint of direct democracy. They are evaluated with regard to their match-up with the public opinion on the key issues from the candidate (party) manifestos, declarations in medias, etc. Next, statistical conclusions about their representative capacity can be made. The statistical test for individual candidates (parties) is accurate, whereas for coalitions it is simplified: It is assumed that, on each question at issue, the position of a coalition is made by the majority rule within the coalition, and in the case of a tied opinion, the prevailing public opinion is taken.

For illustration, five leading German parties and their coalitions are evaluated. For this purpose, the positions of the parties on over 30 topical issues, as given for the last Bundestag (parliament) elections 2009, are compared with results of public opinion polls. The outcomes are summarized in the party indices universality (percentage of issues with majority representation). The same is done for party coalitions. A statistical test is developed to judge whether the index magnitudes are sufficiently high to confirm the representative capacity. It is shown that the representativeness of German parties and their coalitions is statistically insignificant.

## References


[1] A. De Acosta, Moderate deviations and associated Laplace approximations for sums of independent random vectors. *Transactions of the American Mathematical Society*, 1992, 329(1), 357–375.

[2] Ph Barbe and M. Broniatowski, On Sharp Large Deviations for Sums of Random Vectors and Multidimensional Laplace Approximation. *Theory of Probability and Its Applications*, 2005, 49, 561–588.

[3] E. Bolthausen, Laplace approximations for sums of independent random vectors. *Probability Theory and Related Fields*, 1987, 76(2), 167–206.

[4] Bundeszentrale der politische Bildung, *Wahl-O-Mat*, 2010. http://www.bpb.de/methodik/XQJYR3

[5] E.G.Jr. Coffman and G.S. Luecker, *Probabilistic Analysis of Packing and Partitioning Algorithms*. New York: Wiley, 1991.

[6] M.R. Garey, R.L. Graham and D.S. Johnson, Resource constrained scheduling as generalized bin packing. *Journal of Combinatorial Theory (A)*, 1976, 21, 257–298.

[7] M.L. Eaton, *Multivariate Statistics: A Vector Space Approach.* Beachwood, Ohio, USA: Institute of Mathematical Statistics, 2007.

[8] A. Edelman and R.J. Rao, Random matrix theory. *Acta Numerica*, 2005, 1–65.

[9] D. Kendrick, *Stochastic Control for Economic Models*. New York: McGraw-Hill, 1981.

[10] M.L. Mehta, *Random Matrices*. Amsterdam: Elsevier/Academic Press, 2004.

[11] M. Stanke, Sequential selection of random vectors under a sum constraint. *Applied Probability Trust*, 2003, 17 July 2003.

[12] StemWijzer. Wikipedia (NL), 2012. http://nl.wikipedia.org/wiki/StemWijzer

[13] A. Tangian, Selecting predictors for traffic control by methods of the mathematical theory of democracy. *European Journal of Operational Research*, 2007, 181, 986–1003.

[14] A. Tangian, A mathematical model of Athenian democracy. *Social Choice and Welfare*, 2008, 31(4), 537–572.

[15] A. Tangian, Predicting DAX trends from Dow Jones data by methods of the mathematical theory of democracy. *European Journal of Operational Research*, 2008, 185, 1632–1662.

[16] A. Tangian, *Statistical Test for the Mathematical Theory of Democracy.* Düsseldorf: WSI-Diskussionspapier 179, 2012.

[17] A. Tangian, German parliamentary elections 2009 from the viewpoint of direct democracy. *Social Choice and Welfare*, 2013 (forthcoming; available online since 24.01.2012).

[18] A. Tangian, *Mathematical Theory of Democracy.* Berlin–Heidelberg: Springer, 2013.

[19] Wahl-O-Mat. Wikipedia (DE), 2012. http://de.wikipedia.org/wiki/Wahl-O-Mat


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