Minimization of Branching in the Optical Trees with Constraints on the Degree of Nodes

Massinissa Merabet
LIRMM
University Montpellier 2
Montpellier, France
massinissa.merabet@lirmm.fr

Sylvain Durand
LIRMM
University Montpellier 3
Montpellier, France
sylvain.durand@lirmm.fr

Miklos Molnar
LIRMM
University Montpellier 2
Montpellier, France
miklos.molnar@lirmm.fr

***Abstract*—Multicast routing applied to optical networks provide several research problems on spanning tree. In optical networks, the ability of dividing the light signal is limited. Two recently problems try to take into account this constraint: looking for spanning trees with minimum number of branching vertices (vertices of degree strictly greater than 2) and looking for spanning trees such that the sum of branch vertices degrees is minimal. There are two kinds of optical nodes: nodes equipped with splitters, able to divide the input light signal, and nodes without splitters, unable to split the signal. The two problems mentioned above do not distinguish between the type of nodes. In this study, we discuss the relationship between the two problems, we thus prove that the two previous problems are not necessarily linked. We also propose two variants of them, taking into account this additional constraint in the construction of the spanning tree, and we find an experimental upper bound on the number of nodes to equip with splitters in an optical network.**

Keywords—optical network; multicast routing; spars splitters; degree bounded spanning tree.

I. INTRODUCTION

Wavelength-Division Multiplexing (WDM) is an effective technique to exploit the large bandwidth of optical fiber to meet the explosive growth of bandwidth demand in the Internet [1].

Multicast consists in simultaneously transmit information from one source to multiple destinations [1] in a bandwidth efficient way (it duplicates the information only when necessary). From the computational point of view, multicast routing protocols are mainly based on spanning tree structure. When the cost of communications has to be minimized, finding such a tree is NP-complete [2] and is known as the Steiner problem. However, the classical Steiner problem does not take into account the physical constraints of the network needed to perform successfully the multicast routing. Indeed, in order to divide the light signal, some nodes must be equipped with optical splitters. In the optical networks, a node which has the ability to replicating any input signal on any wavelength to any subset of output fibers is referred to as a Multicast-Capable (MC) node [3]. On the other hand, a node which has the ability to tap into the signal and forward it to only one output is called a Multicast-Incapable (MI) node [3]. Optical networks will have a limited number of MC nodes, and these nodes should be positioned such as the multicast routing is feasible. In addition to that, the light power in optical networks should be controlled because of the power loss. Indeed if a light signal is splitted into m copies, the signal power of one copy will be reduced with a factor of $1/m$ of the original signal power [4]. For this reason, it is useful to find a spanning tree such that the number of branching nodes (nodes of degree strictly greater than 2) is limited [5]. To better take into account this constraints it is necessary to find a spanning tree such that the sum of the degrees of nodes dividing the light signal is limited.

Although the two previous problems aim at satisfying reel constraints, they do not take into consideration the ability of an optical node to divide the light signal. They consider that all nodes can be branching nodes in the spanning tree. furthermore, in the examples given in the literature, often the same optimal spanning tree is used for both the first and the second problem. In this study, we introduce two variants of the previous problems that take into consideration the type of an optical node (MC or MI), so that all nodes connecting the spanning tree are effectively able to divide the light signal, and we prove that the two previous problems are not necessarily linked.

The rest of the paper is organized as follows. Section II contains basic definitions and formal statements of the problems considered in this paper. Section III proves that the problems MBV and MDC are not necessary linked. Section IV provides the ILP formulations of MBV-DC and MDS-DC. In Section IV, we analyse the experimental results about MBV-DC and MDS-DC on a set of scenarios. In that Section, we also found an experimental upper bound on the number of nodes to equip with splitters in an optical network. Conclusions are object of Section VI.

II. DEFINITIONS AND FORMULATIONS

Let the topology of an optical network be modelized by a connected graph $G = (V, E)$, where V is set of the vertices (corresponding to optical nodes) and E the set of edges (corresponding to optical links). For each vertex $v \in V$ we denote by $d_G(v)$ the degree of v in G. We denote by $CC(G)$ the number of connected components of the graph G. We denote by $MC(G)$ the set of multicast-capable vertices in the

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graph G and $MI(G)$ the set of multicast-incapable vertices. Let $T = (V_T, E_T)$ be a spanning tree of G. A vertex $v \in V_T$ is a branch vertex in T iff $d_T(v) > 2$. Let $NB(T)$ be the set of branching vertices of the tree T. We denote by $s(T)$ the size of $NB(T)$ and by $q(T)$ sum of the of branching nodes degrees of the tree T ($q(T) = \sum_{v \in NB(T)} d_T(v)$).

We denote by $s^*(G)$ the smallest number of branching nodes of all spanning trees of G and $q^*(G)$ the smallest sum of branching nodes degrees of all spanning trees of G.

The two problems initially proposed in [5] have been defined as follows:

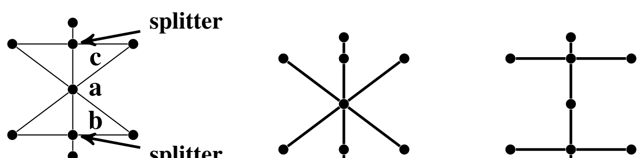
Problem II.1. *The problem MBV (Minimum Branch Vertices spanning tree) consists in finding a spanning tree of G which has the minimum number of branch vertices.*

Problem II.2. *The problem MDS (Minimum Degrees Sum branch vertices spanning tree) consists in finding a spanning tree of G which has the minimum sum of branching nodes degrees.*

We propose the modification of MDS and MBV, such that they take into account the additional constraint of ability of an optical node to divide the light signal. These two new problems allow a network node to be a branch node in the corresponding spanning tree if and only if this node is multicast-capable.

Problem II.3. *The problem MBV-DC (minimum branch vertices spanning tree with degree contraints) consists in finding a spanning tree T of G which has the minimum number of branch vertices such that $NB(T) \subseteq MC(G)$.*

Problem II.4. *The problem MDS-DC (minimum degrees sum of branch vertices spanning tree with degree contraints) consists in finding a spanning tree T of G which has the minimum sum of branch vertices degrees, such that $NB(T) \subseteq MC(G)$.*



(a) **Starting network** (b) **Optimal spanning tree for the MBV** (c) **Optimal spanning tree for the MBV-DC**

Figure 1. Example of the difference between the MBV and MBV-DC

Suppose that the network shown in Figure $1(a)$ contains two multicast-capable nodes : $MC(G) = \{b, c\}$. The optimal solution (Figure $1(b)$) for the problem MBV does not take into account this constraint and selects the node a as a branching node. This tree is not feasible in the optical network. On contrary the optimal solution (Figure $1(c)$) for the problem MBV-DC is greater (two branching nodes) but feasible.

III. Relation between MBV and MDS

In all examples shown in the literature, there is an optimal spanning tree for both the MBV and the MDS. However, the MBV and the MDS are two different problems. In this section, we present an example where the set of optimal spanning trees for MBV and the set of optimal spanning trees for MDS are disjoint.

Remember that $s(T)$ denotes the number of branching vertices of the tree T and $q(T)$ the sum of branching nodes degrees of T. We denote by $s^*(G)$ the smallest number of branching nodes of all spanning trees of G and $q^*(G)$ the smallest sum of the degrees of branching nodes of all spanning trees of G.

Proposition III.1. *The MDS problem and MBV are not linked: There exists a graph G such that:*
For all spanning tree T of G:

1) *If T is optimal for the MBV problem, it is not optimal for the MDS.*
   That is: if $s(T) = s^*(G)$ then $q(T) \neq q^*(G)$,
2) *If T is optimal for the MDS problem, it is not optimal for the MBV.*
   That is: if $q(T) = q^*(G)$ then $s(T) \neq s^*(G)$.

Proof: Figure 2 presents a graph $G = (V, E)$ which respects conditions of Proposition III.1:

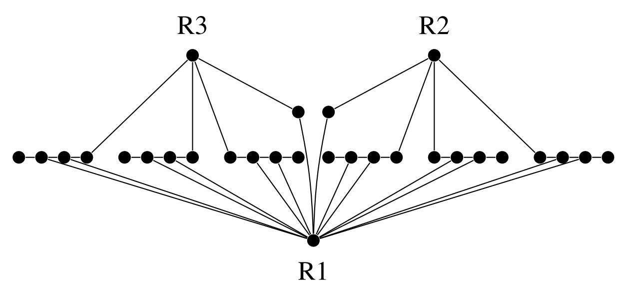


Figure 2. Instance proving the proposition III.1

If G in Figure 2 is Hamiltonian, then any optimal solution for one problem will also be an optimal one for the other one since $s^*(G) = q^*(G) = 0$. Thus, we must first prove that G does not contain Hamiltonian path. We use the following result of [6]:

Proposition III.2. *[6] Let $G(V, E)$ be a graph, if G has a Hamiltonian path, then for all $S \subseteq V$, the graph $(G - S)$ has at most $\mid S + 1 \mid$ connected components.*

Using the contrapositive of proposition III.2 on G with $S = \{R1, R2, R3\}$, we conclude that G is not Hamiltonian.

Thus we have $s^*(G) \geq 1$. The tree $T1$ in Figure 3 is a spanning tree of the graph G and $s(T1) = 1$. Therefore $s^*(G) = 1$.

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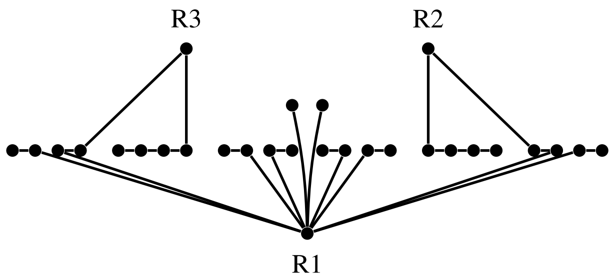

Figure 3. Spanning tree $T1$ of G

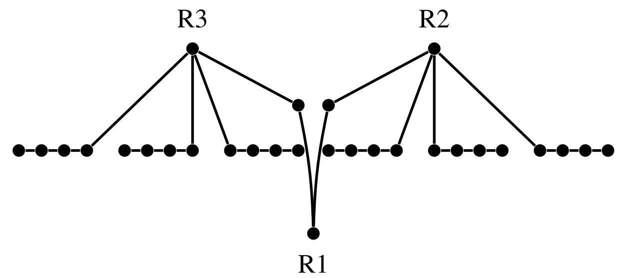

Figure 4. Spanning tree $T2$ of G

■

Let G' be the sub graph of G induced by $V - \{R1, R2, R3\}$. G' is composed of 8 connected components. We conclude that for all spanning tree T of G, T contains at least $R1$ or $R2$ or $R3$ as a branching node. Note that for all spanning tree T of G such that $s(T) = s^*(G) = 1$, $R1$ is the only possible branching node.

We now prove that $q^*(G) = 8$: G is not Hamiltonian so $q^*(G) \geq 3$. In the spanning tree $T2$ of Figure 4 we have $q(T2) = 8$, therefore $8 \geq q^*(G) \geq 3$.

Suppose that $q^*(G) < 6$. Let T be a spanning tree such that $q(T) < 6$. T has a single branching node. But, if $d_T(R1) < 6$ then $CC(T) \geq 2$. So T has at least two branching nodes, which is in contradiction with the hypothesis $q^*(G) < 6$. Therefore, $8 \geq q^*(G) \geq 6$.

Let T be a spanning tree of G such that $8 \geq q(T) \geq 6$, $q(T) = q^*(G)$, T contains 2 branching nodes, and at least $R1$ or $R2$ or $R3$ are branching nodes in T.

If $R1$ is a branching node in T, then there must be at least two other branching nodes so that T is connected. So $R1$ is not a branching node in T (otherwise $q(T) \geq 9$).

Since $d_G(R2) = 4$, $R2$ is a branching node in T, then only $R3$ has a large enough degree in G so that T is connected ($R1$ is already eliminated). Thus $R2$ and $R3$ are the only branching nodes in T.
Symmetrically if $R3$ is a branching node, $R2$ must be the only other one.

For all spanning tree T of G with $R3$ and $R2$ as the only branching nodes, we must have $d_T(R2) = d_G(R2) = 4$ and $d_T(R3) = d_G(R3) = 4$, which implies that $q^*(G) = 8$.

Conclusion: For all spanning tree T of G such that $q(T) = q^*(G) = 8$, $s(T) > 1$, so $s(T) \neq s^*(G)$. For all spanning tree T of G such that $s(T) = s^*(G) = 1$, $q(R1) > 8$, so $q(T) \neq q^*(G)$.

IV. ILP FORMULATION

In this section, we resume from [5] the ILP formulations of MBV and MDS problems, and we modify them in order to take into account the capacity of an optical node to divide the input light signal.

In order to define a spanning tree T of G, we can send from a source vertex $S \in V$ one flow unit to every other vertices $v \in V \setminus \{S\}$. Although edges of G are undirected, we define two variables for each edge $e = \{u, v\} \in E$: f_{uv} and f_{vu} define respectively the flow going from u to v and the flow going from v to u along $\{u, v\}$. For each edge $e = \{u, v\} \in E$, we consider a binary decision variable x_e such that $x_e = 1$ when e belongs to T and $x_e = 0$ otherwise. Finally, for each $v \in V$, we have a decision variable y_v that is equal to 1 if v is a branching node, and 0 otherwise.

Let us denote by $\omega(v) = \{w \in V \mid \{v, w\} \in E\}$ the set of neighbours of v. The mathematical formulation of MBV given in [5] is the following:

$$\begin{cases} min\ s^* = \sum_{v \in V} y_v & \text{(1a)} \\ \sum_{e \in E} x_e = n - 1 & \text{(1b)} \\ \sum_{v \in \omega(S)} f_{Sv} - \sum_{v \in \omega(S)} f_{vS} = n - 1 & \text{(1c)} \\ \sum_{u \in \omega(v)} f_{vu} - \sum_{u \in \omega(v)} f_{uv} = -1, \quad \forall v \in V \setminus \{S\} & \text{(1d)} \\ f_{uv} \leq (n-1)x_e, \quad \forall e = \{u, v\} \in E & \text{(1e)} \\ f_{vu} \leq (n-1)x_e, \quad \forall e = \{u, v\} \in E & \text{(1f)} \\ \sum_{e=(u,v) \mid u \in \omega(v)} x_e - 2 \leq (n-1)y_v,\ \forall v \in V & \text{(1g)} \\ x_e \in \{0, 1\}, \quad \forall e \in E & \text{(1h)} \\ y_v \in \{0, 1\}, \quad \forall v \in V & \text{(1i)} \\ f_{uv} \geq 0, \quad \forall e = \{u, v\} \in E & \text{(1j)} \\ f_{vu} \geq 0, \quad \forall e = \{u, v\} \in E & \text{(1k)} \end{cases}$$

The mathematical model for MDS [5] requires additional integer decision variables counting the degree of branch vertices in the solution:

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$$z_v = \begin{cases} d_T(v), & \text{if } v \text{ is a branching node,} \\ 0, & \text{otherwise.} \end{cases}$$

The objective function is then:

$$min\ q^* = \sum_{v \in V} z_v$$

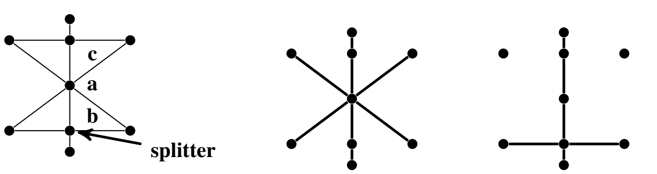
There is an additional constraint:

$$\sum_{e=(u,v)|u \in \omega(v)} x_e - 2 + y_v \leq z_v \qquad \forall v \in V$$

In our problems, we want to satisfy optical constraints imposed by the presence / absence of splitters in nodes. The mathematical formulation of the MBV-DC, resp. MDS-DC, is the same as the MBV, respectively MDS, but we must add the following constraint:

$$y_v = 0 \text{ if } v \notin MC(G)$$

An important difference between the two problems has to be analyzed. For all undirected connected graph input of MBV and MDS, it is guaranteed to have a feasible solution (every connected graph admits a spanning tree). On the contrary, the existence of a feasible solution for the MBV-DC and DC-MDS depends strongly on the positioning of splitters in the network. In Figure 5, only vertex b has a splitter ($MC(G) = \{b\}$), this instance does not have a feasible solution.



(a) **Starting network** (b) **Optimal spanning tree of MBV** (c) **No feasible solution for the MBV-DC**

Figure 5. Example of instance for which there are no feasible solutions of MBV-DC

V. EXPERIMENTAL RESULTS

We measured solutions of MBV-DC (and MDS-DC) according to the proportion nbv of nodes (of degree strictly greater than 2) equipped with splitters in the network. When the proportion is 100% the solution is the same as for MBV (and MDS).

Instances of MBV-DC and MDS-DC are undirected and connected graphs. To produce such graphs, the NetGen random graph generator was used. NetGen is a powerful tool dedicated specifically to the generation of random transport networks [7]. NetGen is used in most experiments on the MBV and MDS already done (especially in [5]). If parameters dedicated to capacities of arcs are set to zero, the generator will produce non-valued connected random graphs. The input files used by NetGen to generate instances follow the format given in Table 1. According to the table, the only parameters that can vary are the seed for the random number generator and the number of vertices and edges of the output graph.

TABLE I. NETGEN PARAMETERS FOR INPUT FILES

parameters	Input	Parameter description
SEED	variable	Random numbers seed
NODES	variable	Number of nodes
SOURCES	1	Number of sources (including transshipment)
SINKS	1	Number of sinks (including transshipment)
DENSITY	variable	Number of (requested) edges
MINCOST	0	Minimum cost of edges
MAXCOST	0	Maximum cost of edges
SUPPLY	1	Total supply
TSOURCES	0	Transshipment sources
TSINKS	0	Transshipment sinks
HICOST	0	Percent of skeleton edges given maximum cost
CAPACITED	0	Percent of edges to be capacitated
MINCAP	0	Minimum capacity for capacitated edges
MAXCAP	0	Maximum capacity for capacitated edges

In order to solve the problems MBV-DC and MDS-DC, we used the linear program solver GLPK [8]. We consider five different values for the number of vertices of random graph: $n \in \{20,\ 30,\ 40,\ 50\}$. For each value of n, we consider a single density value (ratio between the number of edges and the number of vertices) d = 1.5. We have chosen this density because it allows to have a significant number of branching nodes in the solutions. This makes the comparison between the MBV (resp. MDS), and MBV-DC (resp. MDS-DC) be more relevant. We consider seven values for the percentage of nodes equipped with splitters among the nodes of degree strictly greater than 2: $nbv \in \{10\%,\ 15\%,\ 20\%,\ 25\%,\ 30\%,\ 35\%,\ 100\%\}$. If a node has degree smaller or equal to 2, it can not be a branching node whatever the constraints.

A random graph associated with a fixed number of vertices, and a fixed percentage of nodes equipped with splitters is called scenario. In order to have a set of significant test, thirty instances are generated for each scenario.

To analyse results in a meaningful way, it is imperative to consider the percentage of infeasible instances for a given scenario. Note that, if an instance is infeasible for MBV-DC then it is infeasible for MDS-DC, and conversely. Therefore, the proportion of infeasible instances is the same for both problems. We consider that, if this proportion is strictly greater than 50% then the value of MBV-DC and MDS-DC on this scenario is not significant. The Figure 6(a) shows the proportion of infeasible instances for MBV-DC. The curves representing $nbv = 10\%$ et $nbv = 15\%$ are above the threshold of 50%. We therefore consider that the comparison between MBV (resp. MDS) and MBV-DC (resp. MDS-DC) is not significant for $nbv < 20\%$.

Figures 6(b) and 6(c) represent average values of solutions of k feasible instances generated for each scenario, such that k less or equal to 30. Note that if nbv is high, then it approaches the solution of problems without constraints (MBV or MDS), which is represented by $nbv = 100\%$.

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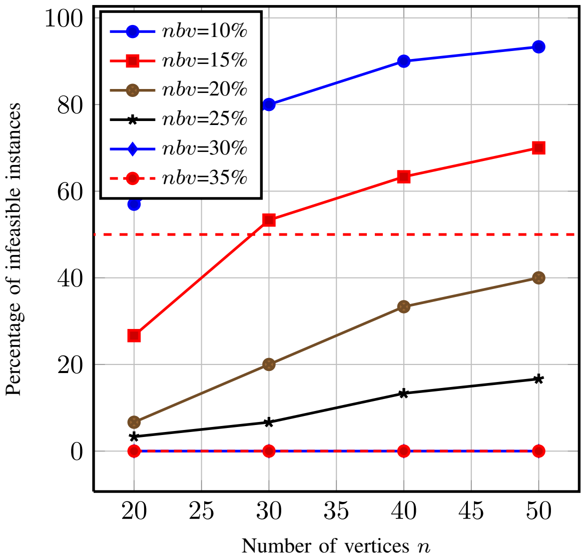
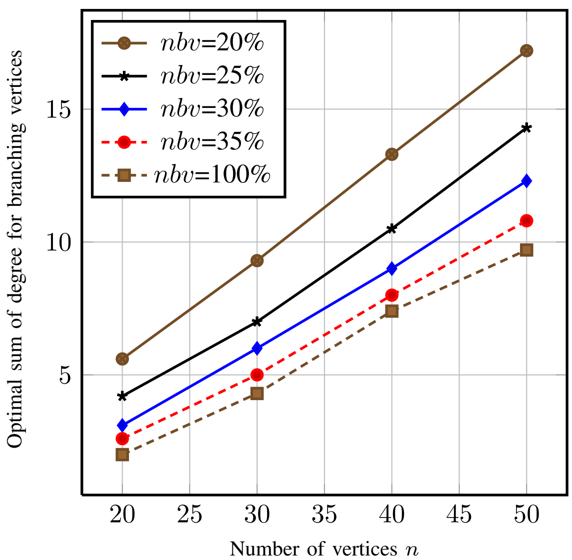
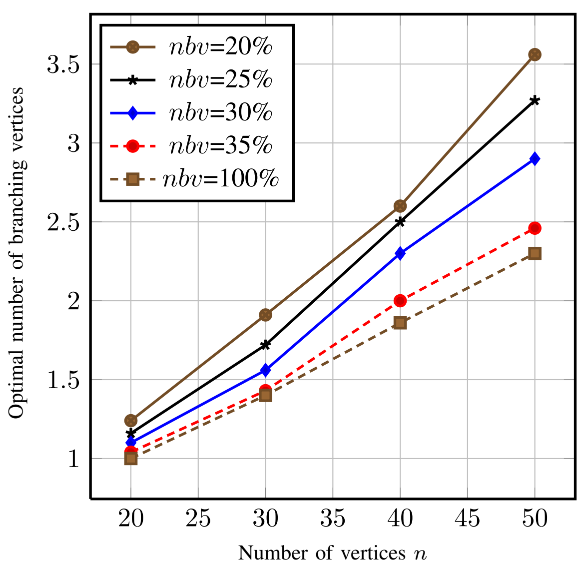


(a) Proportion of infeasible instances

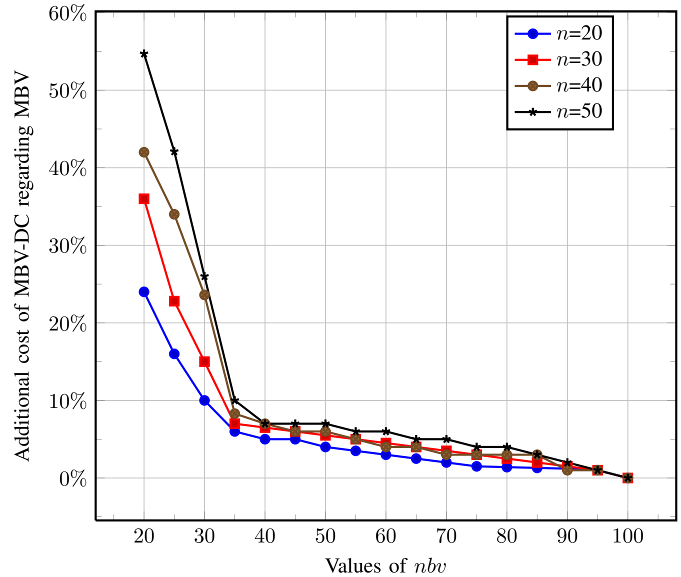


(b) Results for MDS-DC

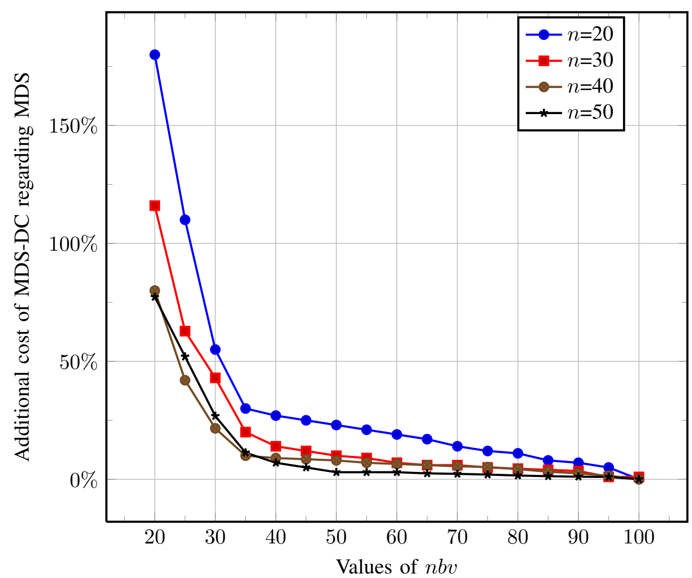


(c) Results for MBV-DC

Figure 6. Results of MBV-DC and MDS-DC



(a) Variation of ratio between MBV-DC and MBV



(b) Variation of ratio between MDS-DC and MDS

Figure 7. Comparison between solutions with or without degree constraints

Moreover, if nbv is high then the percentage of infeasible instances is low. We also observe that, from $nbv = 35\%$, the solution of MBV-DC significantly approach the solution of MBV. For $nbv \geqslant 30\%$, we see that the percentage of infeasible instances is equal to zero.

In Figure 7, we show the influence of the degree constraint for the two studied problems, the percentage of additional cost due to the degree constraint regarding the value of nbv is given for different sizes of networks.

The threshold $nbv = 35\%$ can be considered as an experimental bound about constraints on degrees of nodes problems MBV-DC and MDS-DC. Beyond $nbv = 35\%$,

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this constraint has little impact on the optimal solution of MBV-DC and MDS-DC: when more than 35% of the nodes of degree higher than 2 are randomly designed as MC nodes, the cost of MBV-DC solution is only 10% larger than the cost of the MBV solution. This result is also true for MDS providing that the size of the network is large enough (more than 40 nodes).

The interest of this result in practice is important: From 35% of nodes equipped with splitters, the constraint on the number of nodes equipped with splitters has little effect on the value of the optimal solution. Specially, through this bound we can say that in an optical network, we can position the splitters randomly on 35% of nodes of degree strictly greater than 2, and have a high probability of ensuring that the cost of multicast connection will be weakly influenced.

Note that for MBV-DC (and MDS-DC), the feasibility of an instance can not however be guaranteed only by the proportion of MC nodes. There are infeasible instances such that only one single node is not equipped with splitter (see Figure 8).

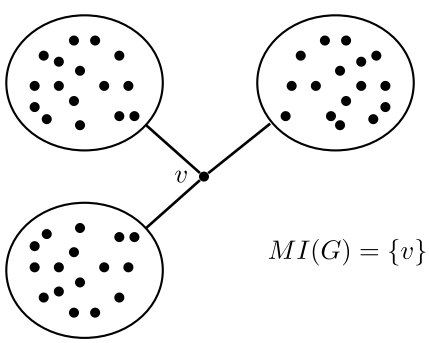

Figure 8. Graph G containing only one single node multicast-incapable, but no feasible solution.

VI. CONCLUSION AND FUTURE WORKS

Two problems have been the source of our study. MBV, which practical interest is to minimize the number of splitters in an optical network, but does not limit the degree of optical nodes, and MDS which practical interest is to minimize the sum of degrees of these splitters nodes in the solution. Both problems consider that all nodes of a network are equipped with optical splitters, and can therefore make divisions of light, which is not true in practice. Indeed, in an optical network, only a subset of the nodes is equipped with splitters (MC nodes). Therefore, only the MC nodes are able to duplicate the light, and to be branching nodes in the spanning tree corresponding to the network, while the other nodes (MI nodes) may only crossed or reached.

The respect of this requirement, it is essential that these theoretical issues best reflect the reality of optical networks. This is why we have introduced two variants of the two problems (problems MBV-DC and MDS-DC) which take into account this constraint in the construction of the spanning tree. Following the resolution of these problems by integer linear programming, and tests on random graphs, we found an experimental upper bound on the number of nodes to equip with splitters in an optical network. Over 35% of nodes equipped with splitters, this constraint has little effect on the corresponding optimal spanning tree. Indeed, beyond this threshold the additional cost due to the degree constraint is less than 10% for the problem MBV-DC. This assumption is also true for MDS-DC provided that the number of nodes is greater than 40.

In problems treated, there is no real limit on the degree of branching nodes because we consider that their degree can be as large as needed in the optimal tree, thus the degree constraint on the nodes in a spanning tree is either 2 (MI nodes) or its degree in the original graph (MC nodes). Knowing that splitters has limited capacity to divide the light signal, consideration may be given to improve the modelling of our problems by setting an upper bound on the degree of the nodes of the spanning tree varying between 1 and the degree of the node in the original graph.

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