Band-Pass Filters for Direct Sampling Receivers

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***Abstract*—A robust analytical design procedure for high performance digital equiripple band-pass finite impulse response filters for direct sampling receivers is introduced. The filters are optimal in Chebyshev sense. The underlaying generating function of the equiripple approximation is the Zolotarev polynomial. The closed form solution provides a straightforward evaluation of the filter degree and of the impulse response coefficients from the filter specification. One example is included. The robustness of the design procedure is emphasized.**

***Keywords*-FIR filter; band-pass filter; equiripple approximation; Zolotarev polynomial; direct sampling; digital receiver;**

I. Introduction

Direct sampling receivers are based on the sampling and processing of the amplified radio-frequency (RF) signal incomming from the aerial. The selectivity in the RF signal is obtained using narrow-band band-pass (BP) digital filters. Because of the high ratio between the pass-band frequency and the bandwidth of the filters, high performance digital filters are required. Such filters can be used in the receivers with direct intermediate frequency (IF) sampling and in frequency analyzers as well. Because of the inherent stability and because of the linear phase the digital finite impulse response (FIR) filters are preferred. A filter is optimal in terms of its length provided its frequency response exhibits an equiripple (ER) behavior. In [1] we have introduced an analytical design procedure for the digital ER notch FIR filters. Here, we present an analytical design procedure for the ER BP FIR filters. The proposed design procedure is based on Zolotarev polynomials [2]-[5]. We present here the closed form solution for the design of ER BP FIR filters. It includes the degree equation and formulas for the robust evaluation of the impulse response coefficients of the ER BP FIR filter.

II. Zero Phase Transfer Function

We assume the impulse response $h(k)$ with odd length $N = 2n+1$ with even symmetry

$$a(0) = h(n)\ ,\ a(k) = 2h(n-k) = 2h(n+k)\ ,\ k = 1 \dots n\ . \tag{1}$$

The transfer function of the filter is

$$\begin{aligned} H(z) &= \sum_{k=0}^{2n} h(k)\, z^{-k} \\ &= z^{-n}\left[h(n) + 2\sum_{k=1}^{n} h(n\pm k)\,\frac{1}{2}\left(z^{k}+z^{-k}\right)\right] \\ &= z^{-n}\sum_{k=0}^{n} a(k)\, T_k(w) = z^{-n} Q(w) \end{aligned} \tag{2}$$

where

$$T_k(w) = \cos(k \arccos(w)) \tag{3}$$

is Chebyshev polynomials of the first kind. The function

$$Q(w) = \sum_{k=0}^{n} a(k)\, T_k(w) \tag{4}$$

represents a polynomial in the variable

$$w = \frac{1}{2}(z + z^{-1}) \tag{5}$$

which on the unit circle $z = e^{j\omega T}$ reduces to the real valued zero phase transfer function (ZPTF) $Q(w)$ of the real argument

$$w = \cos(\omega T)\ . \tag{6}$$

III. Generating Polynomial

An approximation of the frequency response of a filter is based on the generating function. The generating function of an ER BP FIR filter is the Zolotarev polynomial $Z_{p,q}(w,\kappa)$ which approximates a constant value in equiripple Chebyshev sense in two disjoint intervals $(-1, w_1)$ and $(w_2, 1)$ as shown in Fig. 1. The lobe with the maximal value $y_m = Z_{p,q}(w_m,\kappa)$ is located inside the interval (w_1, w_2). The notation $Z_{p,q}(w,\kappa)$ emphasizes the fact that the integer value p counts the number of zeros right from the maximum w_m and the integer value q corresponds to the number of zeros left from the maximum w_m. The real value $0 \le \kappa \le 1$ which is in fact the Jacobi elliptical modulus affects the maximum value y_m and the width $w_2 - w_1$ of this lobe (Fig. 1). For increasing κ the value y_m increases and the lobe broadens. E. I. Zolotarev

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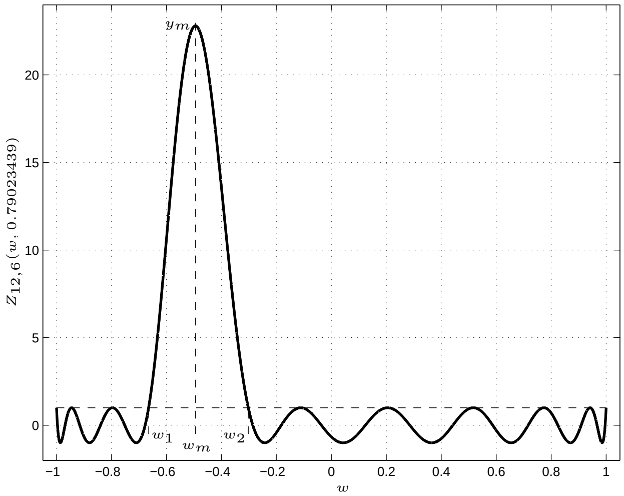

Fig. 1. Zolotarev polynomial $Z_{12,6}(w, 0.79023439)$.

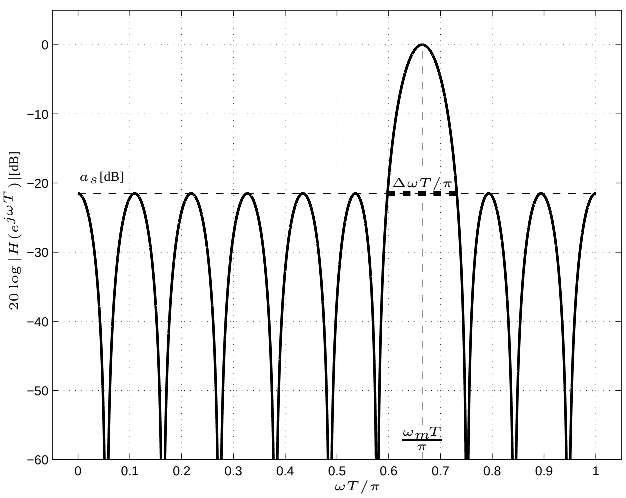

Fig. 2. Amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB] corresponding to the Zolotarev polynomial from Fig. 1.

(1847-1878) derived the general solution of this approximation problem in terms of Jacobi elliptic functions [3]-[5]

$$Z_{p,q}(w,\kappa) = \frac{(-1)^p}{2} \tag{7}$$
$$\times \left[\left(\frac{H(u - \frac{p}{n}\ \mathbf{K}(\kappa))}{H(u + \frac{p}{n}\ \mathbf{K}(\kappa))} \right)^n + \left(\frac{H(u + \frac{p}{n}\ \mathbf{K}(\kappa))}{H(u - \frac{p}{n}\ \mathbf{K}(\kappa))} \right)^n \right] \ .$$

The factor $(-1)^p/2$ appears in (7) as the Zolotarev polynomial alternates $(p+1)-$times in the interval $(w_2, 1)$. The variable u is expressed by the incomplete elliptical integral of the first kind

$$u = F\left(\mathrm{sn}\left(\frac{p}{n}\ \mathbf{K}(\kappa)|\kappa\right) \sqrt{\frac{1+w}{w + 2\,\mathrm{sn}^2\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right) - 1}} |\kappa \right) \ . \tag{8}$$

The function $H\left(u \pm (p/n)\ \mathbf{K}(\kappa)\right)$ is the Jacobi Eta function, $\mathrm{sn}(u|\kappa)$, $\mathrm{cn}(u|\kappa)$, $\mathrm{dn}(u|\kappa)$ are Jacobi elliptic functions, $\mathbf{K}(\kappa)$ is the quarter-period given by the complete elliptic integral of the first kind and $F(\phi|\kappa)$ is the incomplete elliptic integral of the first kind. The degree of the Zolotarev polynomial is $n = p + q$. A comprehensive treatise of Zolotarev polynomials was published in [5]. It includes the analytical solution of the coefficients of Zolotarev polynomials, the algebraic evaluation of the Jacobi Zeta function $\mathrm{Z}(\frac{p}{n}\mathbf{K}(\kappa)|\,\kappa)$ and of the elliptic integral of the third kind $\Pi(\sigma_m, \frac{p}{n}\mathbf{K}(\kappa)|\,\kappa)$. The Jacobi Zeta function and the elliptic integral of the third kind are connected by the formula

$$\Pi(u, u_0|\,\kappa) = \frac{1}{2} \ln \frac{\Theta(u - u_0)}{\Theta(u + u_0)} + u\mathrm{Z}(u_0|\,\kappa) \tag{9}$$

where

$$u_0 = \frac{p}{p+q}\ \mathbf{K}(\kappa) \tag{10}$$

and $\Theta(w)$ is the Jacobi Theta function [4]. The position w_m of the maximum value $y_m = Z_{p,q}(w_m, \kappa)$ is

$$w_m = w_1 + 2\,\frac{\mathrm{sn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right) \mathrm{cn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)}{\mathrm{dn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)} \mathrm{Z}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right) \tag{11}$$

where the edges of the lobe are

$$w_1 = 1 - 2\,\mathrm{sn}^2\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right) \tag{12}$$

$$w_2 = 2\,\mathrm{sn}^2\left(\frac{q}{n}\mathbf{K}(\kappa)|\kappa\right) - 1 \ . \tag{13}$$

The relation for the maximum value y_m

$$y_m = \cosh 2n\left(\sigma_m \mathrm{Z}(\frac{p}{n}\mathbf{K}(\kappa)|\kappa) - \Pi(\sigma_m, \frac{p}{n}\mathbf{K}(\kappa)|\kappa)\right) \tag{14}$$

is useful in the normalization of Zolotarev polynomials. The degree of the Zolotarev polynomial $Z_{p,q}(w,\kappa)$ is expressed by the degree equation

$$n \geq \frac{\ln(y_m + \sqrt{y_m^2 - 1})}{2\sigma_m \mathrm{Z}(\frac{p}{n}\mathbf{K}(\kappa)|\kappa) - 2\Pi(\sigma_m, \frac{p}{n}\mathbf{K}(\kappa)|\kappa)} \ . \tag{15}$$

The auxiliary parameter σ_m is given by the formula

$$\sigma_m = F\left(\arcsin\left(\frac{1}{\kappa\ \mathrm{sn}\left(\frac{p}{n}\ \mathbf{K}(\kappa)|\kappa\right)} \sqrt{\frac{w_m - w_s}{w_m + 1}} \right) |\kappa \right) \tag{16}$$

where $F(\Phi|\kappa)$ is the incomplete elliptical integral of the first kind. The Zolotarev polynomial $Z_{p,q}(w,\kappa)$ satisfies the differential equation

$$(1 - w^2)(w - w_1)(w - w_2)\left(\frac{dZ_{p,q}(w,\kappa)}{dw}\right)^2 \tag{17}$$
$$= n^2\left(1 - Z_{p,q}^2(w,\kappa)\right)(w - w_m)^2 \ .$$

The differential equation expresses the fact that the derivative $dZ_{p,q}(w,\kappa)/dw$ does not vanish at the points $w = \pm 1$, w_1, w_2 where $Z_{p,q}(w,\kappa) = \pm 1$ for which the right hand side of eq. (17) vanishes, and that $w = w_m$ is a turning point corresponding to the local extrema at which $Z_{p,q}(w,\kappa) \neq \pm 1$.

Based on the differential equation (17) we have derived the recursive algorithm for the evaluation of the impulse response $h(k)$ corresponding to the Zolotarev polynomial $Z_{p,q}(w,\kappa)$ based on its expansion into Chebyshev polynomials of the first kind

$$Z_{p,q}(w,\kappa) = \sum_{k=0}^{n} a(k) T_k(w) \quad . \tag{18}$$

The corresponding recursive algorithm is summarized in Table I.

IV. Design Procedure

There are two goals in the design of any filter. The first one is to obtain the minimal filter degree n (or minimal filter length N) satisfying the filter specification while the second one is to evaluate the impulse response $h(k)$ of the filter. The ER BP FIR filter is specified by the pass-band frequency $\omega_m T$ and by the bandwidth $\Delta\omega T$ for the attenuation a_s [dB] in the stop-bands (Fig. 2). The proposed design procedure consists of several steps as follows:

1) Specify the pass-band frequency $\omega_m T$ (or f_m), width of the pass-band $\Delta\omega T$ (or Δf) and the attenuation in the stop-bands a_s [dB] (Fig. 2). For the non-normalized frequencies f_m and Δf specify additionally the sampling frequency f_s.
2) Evaluate the normalized frequencies

$$\omega_m T = \pi \frac{f_m}{\frac{f_s}{2}} \quad , \quad \Delta\omega T = \pi \frac{\Delta f}{\frac{f_s}{2}} \quad . \tag{19}$$

   if the filter is specified by the non-normalized ones.
3) Calculate the band edges

$$\omega_2 T = \omega_m T - \frac{\Delta\omega T}{2} \quad , \quad \omega_1 T = \omega_m T + \frac{\Delta\omega T}{2} \quad . \tag{20}$$

4) Evaluate the Jacobi elliptic modulus κ

$$\kappa = \sqrt{1 - \frac{1}{\tan^2(\varphi_1)\tan^2(\varphi_2)}} \tag{21}$$

   for the auxiliary parameters φ_1 and φ_2

$$\varphi_1 = \frac{\omega_1 T}{2} \quad , \quad \varphi_2 = \frac{\pi - \omega_2 T}{2} \quad . \tag{22}$$

5) Calculate the rational values p/n and q/n

$$\frac{p}{n}\mathbf{K}(\kappa) = F(\varphi_1|\kappa) \quad , \quad \frac{q}{n}\mathbf{K}(\kappa) = F(\varphi_2|\kappa) \quad . \tag{23}$$

6) Determine the required maximum value y_m

$$y_m = \frac{2}{10^{0.05 a_s[dB]}} \quad . \tag{24}$$

7) Calculate and round up the minimum degree n required to satisfy the filter specification using the degree equation (15). For the algebraic evaluation of the Jacobi Zeta function $\mathrm{Z}(\frac{p}{n}\mathbf{K}(\kappa)|\,\kappa)$ and of the elliptic integral of the third kind $\Pi(\sigma_m, \frac{p}{n}\mathbf{K}(\kappa)|\,\kappa)$ in the degree equation (15) use the algebraical procedures [5].
8) Calculate the integer values p and q of the Zolotarev polynomial $Z_{p,q}(w,\kappa)$

$$p = \left[n\frac{F(\varphi_1|\kappa)}{\mathbf{K}(\kappa)} \right] \quad , \quad q = \left[n\frac{F(\varphi_2|\kappa)}{\mathbf{K}(\kappa)} \right] \quad . \tag{25}$$

   The brackets [ ] in (25) stand for rounding.
9) For the values p, q, κ and y_m evaluate the impulse reponse $h(k)$ algebraically using the procedure summarized in Tab. I.

V. Example of the Design

Design the ER BP FIR filter specified by the pass-band frequency $f_m = 10.7$ MHz and by the bandwidth $\Delta f = 50$ kHz for minimal attenuation in the stop-band $a_s = -80$ dB. The specified sampling frequency is $f_s = 30$ MHz.

From the filter specification we get $\omega_m T/\pi = 0.71\overline{3}$ and $\Delta\omega T/\pi = 0.00\overline{3}$ (19). Further we get $\kappa = 0.16239149$ (21), $n = 2026$ (15), $p = 1445$ and $q = 581$ (25). The filter length is $N = 4053$ coefficients. The actual attenuation in the stop-bands is $a_{s\,act} = -80.13$ dB. The amplitude frequency response of the ER BP FIR filter is shown in Fig. 3. A detailed view of its passband is shown in Fig. 4.

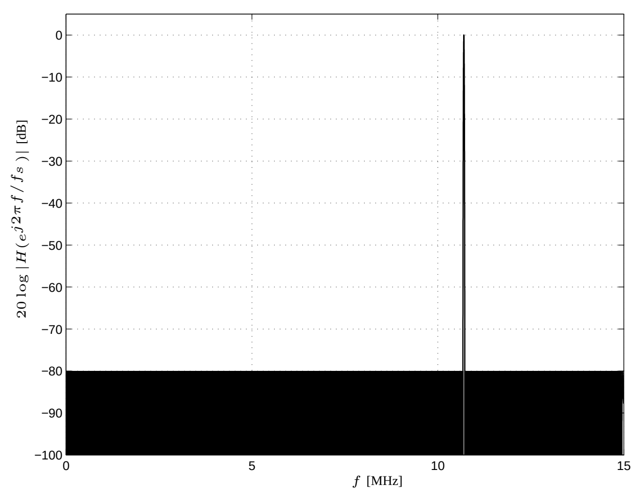

Fig. 3. Amplitude frequency response $20\log|H(e^{j2\pi f/f_s})|$ [dB].

VI. Robustness of the Design Procedure

In order to demonstrate the robustness of the presented design procedure, let us design the filter which was specified in our example, however, with modified bandwidth which is now specified by $\Delta f = 5$ kHz.

We get $\kappa = 0.05166139$ (21), $n = 20248$ (15), $p = 14444$ and $q = 5804$ (25). The filter length is $N = 40497$ coefficients. The actual attenuation in the stop-bands is $a_{s\,act} = -80.04$ dB. The amplitude frequency response of the ER BP FIR filter is shown in Fig. 5. A detailed

TABLE I
ALGORITHM FOR THE EVALUATION OF THE IMPULSE RESPONSE $h(k)$.

given

$p,\ q,\ \kappa,\ y_m$

initialisation

$n = p + q$

$w_1 = 1 - 2\,\mathrm{sn}^2\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)$

$w_2 = 2\,\mathrm{sn}^2\left(\frac{q}{n}\mathbf{K}(\kappa)|\kappa\right) - 1$

$w_a = \frac{w_1 + w_2}{2}$

$w_m = w_1 + 2\,\frac{\mathrm{sn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)\mathrm{cn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)}{\mathrm{dn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)}\mathrm{Z}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)$

$\alpha(n) = 1$

$\alpha(n+1) = \alpha(n+2) = \alpha(n+3) = \alpha(n+4) = \alpha(n+5) = 0$

body

(*for* $m = n + 2$ *to* 3)

$8c(1) = n^2 - (m+3)^2$

$4c(2) = (2m+5)(m+2)(w_m - w_a) + 3w_m[n^2 - (m+2)^2]$

$2c(3) = \frac{3}{4}[n^2 - (m+1)^2] + 3w_m[n^2 w_m - (m+1)^2 w_a] - (m+1)(m+2)(w_1 w_2 - w_m w_a)$

$c(4) = \frac{3}{2}(n^2 - m^2) + m^2(w_m - w_a) + w_m(n^2 w_m^2 - m^2 w_1 w_2)$

$2c(5) = \frac{3}{4}[n^2 - (m-1)^2] + 3w_m[n^2 w_m - (m-1)^2 w_a] - (m-1)(m-2)(w_1 w_2 - w_m w_a)$

$4c(6) = (2m-5)(m-2)(w_m - w_a) + 3w_m[n^2 - (m-2)^2]$

$8c(7) = n^2 - (m-3)^2$

$\alpha(m-3) = \frac{1}{c(7)}\sum_{\mu=1}^{6} c(\mu)\alpha(m+4-\mu)$

(*end loop on* m)

normalisation

$s(n) = \frac{\alpha(0)}{2} + \sum_{m=1}^{n}\alpha(m)$

$a(0) = (-1)^p\,\frac{\alpha(0)}{2s(n)}$

(*for* $m = 1$ *to* n)

$a(m) = (-1)^p\,\frac{\alpha(m)}{s(n)}$

(*end loop on* m)

impulse response

$h(n) = \frac{y_m - a(0)}{y_m + 1}$

(*for* $m = 1$ *to* n)

$h(n \pm m) = -\frac{a(m)}{2(y_m + 1)}$

(*end loop on* m)

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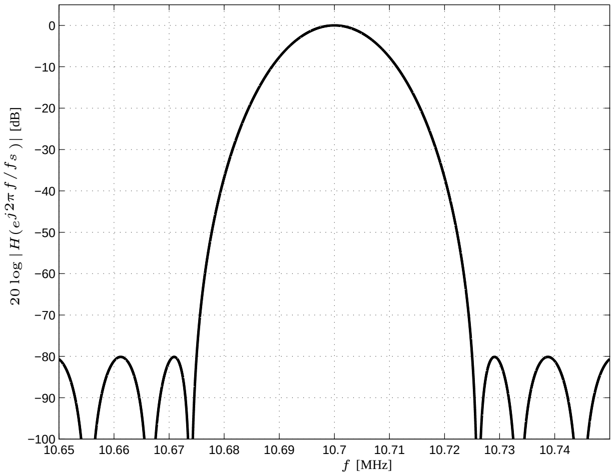

Fig. 4. Detailed view of the pass-band.

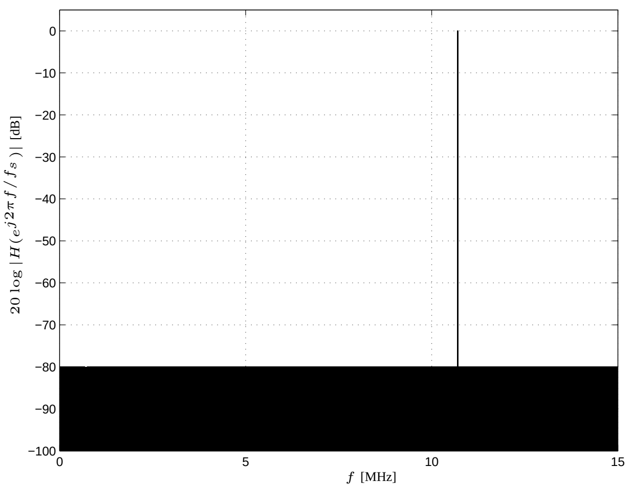

Fig. 5. Amplitude frequency response $20\log|H(e^{j2\pi f/f_s})|$ [dB].

view of its passband is shown in Fig. 6. In order to point out the robustness of our design procedure, let us do comparative designs of both filters presented above by the established numerical Parks-McClellan procedure [6] which is implemented e.g. in the Matlab function *firgr*. The filter with the length of 4053 coefficients from our example can be designed using the function call [h,err,res]=firgr(4052,[0 10.675 10.7 10.725 15]/15,[0 0 1 0 0],'n' 'n' 's' 'n' 'n') easily. Note that the Matlab function $firgr$ returns filters with different normalization of the ripples of the corresponding frequency response. Except for the difference in the normalization, the obtained results are identical. On the other hand the design of the filter with the length of 40497 coefficients specified in this section cannot be designed using the function call [h,err,res]=firgr(40496,[0 10.6975 10.7 10.7025 15]/15,[0 0 1 0 0],'n' 'n' 's' 'n' 'n') as it collapses (Matlab R2010b) because of numerical problems. We assume that this failure is caused by the failed numerical evaluation of the densely located roots (isoextremal values) of the optimized function. To our knowledge, our design

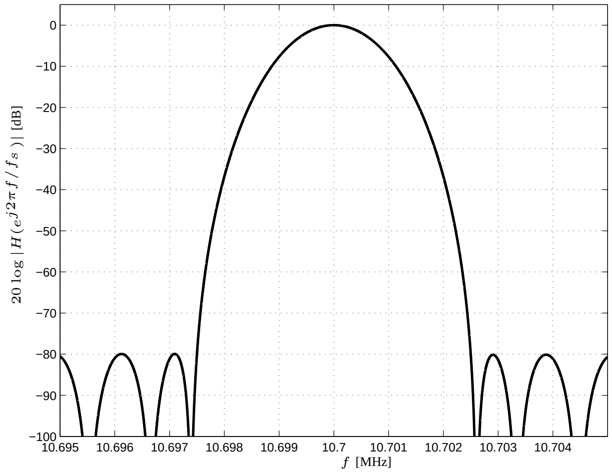

Fig. 6. Detailed view of the pass-band.

procedure presented here has no parallel in the design of high performance ER BP FIR filters with the length beyond several thousands of coefficients. Further note that in the Parks-McClellan design procedure the length of the filter is an input argument, not the result of the design.

VII. Implementation of the Filter

There are various ways for implementing of FIR filters in real time available. For high order filters, the digital signal processors (DSP) and the field programmable gate arrays (FPGA) dominate. We prefer the filter implementation using digital DSPs over the implementation using FPGAs mainly because it is a less time consuming process. The band-pass FIR filter with the length of 4053 coefficients from our example requires 122 billions multiply-and-accumulate (GMACs) operations per second. The adequate DSP is the eight-core DSP TMS320C6678 [7] which provides 320GMACs operations per second in the 16-bit fixed point arithmetics. For the real-time implementation of the filter we use the DSP Evaluation Module TMDXEVM6678 (Fig. 7). The implementation of the filter in the IEEE-754 compliant single precision floating point arithmetics would require a two chip solution based on the DSP TMS320C6678 which provides 160 billions floating point operations (GFLOPs) per second per chip, or a single chip solution based on the 32-core DSP TMS320TCI6609 [8]. The implementation of the filter with the length of 40497 coefficients from previous section requires 1215 GMACs and consequently its implementation would require a multi-chip solution, e.g. three chip solution based on the 32-core DSP TMS320TCI6609.

VIII. Conclusions

We have presented an analytical design of high performance digital equiripple band-pass finite impulse response filters. In contrast to the established numerical design procedures the proposed design method is based on the generating polynomial

Fig. 7. Evaluation Module TMDXEVM6678.

and provides a formula for the degree of the filter and formulas for the evaluation of the coefficients of the impulse response of the filter. The demonstrated robustness is another advantage of the proposed design method.

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REFERENCES

[1] P. Zahradnik, M. Vlček, Fast Analytical Design Algorithms for FIR Notch Filters, *IEEE Transactions on Circuits and Systems*. March 2004 , Vol. 51, No. 3, pp. 608 - 623.
[2] N. I. Achieser, Über einige Funktionen, die in gegebenen Intervallen am wenigsten von Null abweichen, *Bull. de la Soc. Phys. Math. de Kazan*, Vol. 3, pp. 1 - 69, 1928.
[3] D. F. Lawden *Elliptic Functions and Applications* Springer-Verlag, New York Inc., 1989.
[4] M. Abramowitz, I. Stegun, *Handbook of Mathematical Function*, Dover Publication, New York Inc., 1972.
[5] M. Vlček, R. Unbehauen, Zolotarev Polynomials and Optimal FIR Filters, *IEEE Transactions on Signal Processing*, Vol.47, No.3, pp. 717-730, March 1999.
[6] J. H. Mc Clellan, T. W. Parks, L. R. Rabiner, FIR Linear Phase Filter Design Program, *Programs for Digital Signal Processing*. newblock New York: IEEE Press, 1979.
[7] www.ti.com/product/tms320c6678
[8] www.ti.com/lit/ml/sprt619/sprt619.pdf