Reflectionless and Equiscattering Quantum Graphs

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***Abstract*—The inverse scattering problem of a quantum star graph is shown to be solvable as a diagonalization problem of Hermitian unitary matrix when the connection condition is given by scale invariant Tsutsui-Fulop form. This enables the construction of quantum graphs with desired properties in tailor-made fashion. The quantum vertices with uniform and reflectionless scatterings are examined, and their finite graph approximations are constructed.**

***Index Terms*—quantum graph; singular vertex; quantum wire; inverse scattering**

I. Introduction

The inverse scattering is one of the most intriguing problems in quantum mechanics. The interest in the inverse scattering problem of quantum graph [1], [2], [3], in particular, is two-fold. Since the quantum graph is a prime example of nontrivial solvable system [4], it presents a challenge for extending the range of solvable inverse scattering problems. The inverse scattering problem of quantum graph is also important for its relevance as the design principle of nanowire-based single electron devices.

In this article, we consider the inverse scattering problem on a star graph with Fulop-Tsutsui vertices [5], the scale invariant subset of most general vertex couplings [6]. A star graph is the elementary building block of generic graph having many half-lines connected together at a single point, the singular vertex. The scattering matrix of star graph with Fulop-Tsutsui condition is energy independent. We exploit this simplicity to give the full answer to its inverse scattering problem in the form of eigenvalue problem of Hermitian unitary matrix. Two special examples of inverse scattering problems, that of reflectionless transmission, and of equal-scattering including the reflection, are examined. Intriguing designs emerge for the realization of quantum device with such properties. Since any singular vertex is effectively reduced to Fulop-Tsutsui vertex in both high and low energy limits [7], our study hopefully opens up a door for the full study of inverse scattering problems for general singular vertex.

This article is organized as follows: In the second section, we formulate the inverse scattering problem of scale invariant graph vertices in terms of matrix diagonalization. In the third section, a scheme to approximate the vertex with small structures made up of δ-vertices is developed. In the fourth section, the scheme is applied to obtain reflectionless and equitransmitting quantum graphs. The accuracy of the approximating procedure is examined in the fifth section, which is followed by the concluding sixth section.

II. Inverse scattering as diagonalization

Consider a singular quantum vertex of degree n, having n half-lines sticking out of a point-like node. The scale invariant subfamily of most general connection condition is characterized by a complex matrix T of size $(n-m)\times m$ where m can take the integer value $m = 1, 2, ..., n-1$, and is given by

$$\begin{pmatrix} I^{(m)} & T \\ 0 & 0 \end{pmatrix} \Psi' = \begin{pmatrix} 0 & 0 \\ -T^\dagger & I^{(n-m)} \end{pmatrix} \Psi, \tag{1}$$

where $I^{(l)}$ signifies the identity matrix of size $l \times l$, and the boundary vectors Ψ and Ψ' are defined by

$$\Psi = \begin{pmatrix} \psi_1(0) \\ \vdots \\ \psi_n(0) \end{pmatrix}, \quad \Psi' = \begin{pmatrix} \psi'_1(0) \\ \vdots \\ \psi'_n(0) \end{pmatrix}, \tag{2}$$

in which $\psi_i(x_i)$ and $\psi'_i(x_i)$ are the wave function and its derivative on i-th line [8]. The coordinates x_i on the i-th line are labeled outwardly from the singular vertex, which is assigned $x_i = 0$ for all i. To achieve the from (1), we may have to suitably renumber lines, in general. The quantum particle coming in from the j-th line and scattered off the singular vertex is described by the scattering wave function on the i-th line, $\psi_i^{(j)}(x)$ which is given in the form

$$\psi_i^{(j)}(x) = \delta_{i,j} e^{-\mathrm{i}kx} + \mathcal{S}_{i,j} e^{\mathrm{i}kx}. \tag{3}$$

From (1) and (3), we obtain the equation

$$\mathrm{i}k \begin{pmatrix} I^{(m)} & T \\ 0 & 0 \end{pmatrix} (\mathcal{S} - I^{(n)}) = \begin{pmatrix} 0 & 0 \\ -T^\dagger & I^{(n-m)} \end{pmatrix} (\mathcal{S} + I^{(n)}). \tag{4}$$

We easily obtain the explicit solution of the scattering matrix $\mathcal{S} = \{\mathcal{S}_{i,j}\}$ in the form

$$\mathcal{S} = -I^{(n)} + 2 \begin{pmatrix} I^{(m)} \\ T^\dagger \end{pmatrix} \left(I^{(m)} + TT^\dagger \right)^{-1} \left(I^{(m)} \ \ T \right). \tag{5}$$

We can rewrite this solution in the form of a products of three Hermitian matrices as

$$\mathcal{S} = X_m^{-1} Z_m X_m, \tag{6}$$

where we define

$$X_m = \begin{pmatrix} I^{(m)} & T \\ T^\dagger & -I^{(n-m)} \end{pmatrix}, \ Z_m = \begin{pmatrix} I^{(m)} & 0 \\ 0 & -I^{(n-m)} \end{pmatrix}. \tag{7}$$

Note the Hermitian unitarity of $\mathcal{S}$;

$$\mathcal{S}^\dagger = \mathcal{S}, \quad \mathcal{S}^\dagger \mathcal{S} = I^{(n)}. \tag{8}$$

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Interestingly, (6) can also be viewed as the diagonalization of Hermitian unitary matrix $\mathcal{S}$ by a non-unitary Hermitian matrix X_m. We can show, in fact, that this form leads to the path to the inverse scattering problem for quantum graph vertex of Fulop-Tsutsui type: Let us suppose that the full set of scattering data is given in terms of an arbitrary Hermitian unitary matrix $\mathcal{S}$. Let us signify the rank of the matrix $\mathcal{S}+I^{(n)}$ by m. After proper renumbering of lines, we can write this matrix in the form

$$\mathcal{S} + I^{(n)} = \begin{pmatrix} I^{(m)} \\ T^\dagger \end{pmatrix} M \begin{pmatrix} I^{(m)} & T \end{pmatrix}, \tag{9}$$

where M is a Hermitian $m \times m$ matrix, and T, a complex $(n-m) \times m$ matrix. From the unitarity of $\mathcal{S}$, we find the relation $(\mathcal{S} + I^{(n)})^2 = 2(\mathcal{S} + I^{(n)})$, from which we obtain

$$M = 2(I^{(m)} + TT^\dagger)^{-1}, \tag{10}$$

and we therefore arrive at (5). It is notable that any Hermitian unitary matrix can be viewed as a solution $\mathcal{S}$ of a Fulop-Tsutsui vertex, which is necessarily independent of the incoming momentum k. For a quantum star graph to break scale invariance and obtain k-dependence, its scattering matrix has to obtain non-Hermiticity. These observations, along with the very fact of the existence and uniqueness of inverse scattering solution of quantum star graph, can be reached easily and directly from the original "U-form" of connection condition using a unitary matrix [1], [6], but our procedure holds definite advantage of giving us T directly, which is known [8] to allow us the physical construction of a finite quantum graph whose small size limit reproduces the prescribed $\mathcal{S}$.

The procedure of diagonalization, in practice, can be cumbersome for large n. There are simpler alternative to obtain T from $\mathcal{S}$: Let us divide $\mathcal{S}$ into four submatrices $\mathcal{S}_{11}$, $\mathcal{S}_{12}$, $\mathcal{S}_{21}$ and $\mathcal{S}_{22}$ of size $m \times m$, $m \times (n-m)$, $(n-m) \times m$ and $(n-m) \times (n-m)$, respectively as

$$\mathcal{S} = \begin{pmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ \mathcal{S}_{21} & \mathcal{S}_{22} \end{pmatrix}. \tag{11}$$

These submatrices have the properties

$$\mathcal{S}_{11}^\dagger = \mathcal{S}_{11}, \quad \mathcal{S}_{22}^\dagger = \mathcal{S}_{22}, \quad \mathcal{S}_{21}^\dagger = \mathcal{S}_{12}, \tag{12}$$

and also

$$\mathcal{S}_{11}^2 + \mathcal{S}_{12}^2 = I^{(m)}, \quad \mathcal{S}_{22}^2 + \mathcal{S}_{21}^2 = I^{(n-m)}, \quad \mathcal{S}_{11}\mathcal{S}_{12} + \mathcal{S}_{12}\mathcal{S}_{22} = 0. \tag{13}$$

We have the explicit expressions of f T in terms of $\mathcal{S}_{ij}$;

$$T = \left(I^{(m)} + \mathcal{S}_{11}\right)^{-1} \mathcal{S}_{12} = \mathcal{S}_{21}^\dagger \left(I^{(n-m)} - \mathcal{S}_{22}\right)^{-1}. \tag{14}$$

It is easy to check that the forms (5) and (6) can be kept under the index renumbering $\alpha \leftrightarrow \beta$ both for $\alpha, \beta \le m$ and for $\alpha, \beta > m$ with the proper transformation for the elements of T; It is given by $t_{\alpha j} \leftrightarrow t_{\beta j}$ for the former and $t_{i\alpha} \leftrightarrow t_{i\beta}$ for the latter. For the case of $\alpha \le m$ and $\beta > m$, it is given by $t_{ij} \to t'_{ij}$ with

$$t'_{ij} = \frac{t_{ij}t_{\alpha\beta} - t_{\alpha j}t_{i\beta}}{t_{\alpha\beta}}\bar{\delta}_{i\alpha}\bar{\delta}_{j\beta} - \frac{t_{\alpha j}\delta_{i\alpha} - \delta_{\alpha j}t_{i\alpha} + \delta_{i\alpha}\delta_{j\beta}}{t_{\alpha\beta}}, \tag{15}$$

where we define $\bar{\delta}_{ij} = 1 - \delta_{ij}$. This implies that it is not possible to exchnage the indices α and β whose $t_{\alpha\beta}$ is zero. This corresponds to the index ordering for which both $(I^{(m)} + \mathcal{S}_{11})$ and $(I^{(n-m)} - \mathcal{S}_{22})$ are singular and the T is undefined, thus the boundary condition at the singular vertex does not take the form (1).

III. Finite approximation

Finite tubes connected at a node tend, in their small diameter limit, to a vertex with delta-like connections, and very often to its strength zero limit, a free vertex [9]. We might also consider applying localized magnetic field to achieve phase change. It is natural, therefore, to devise a design principle to construct arbitrary connection condition out of this elementary vertex. Once all elements of $T = \{t_{ij}\}$, $i = 1, ..., m$ and $j = m+1, .., n$, are obtained, a finite graph with internal lines and the δ-coupling vertices can be constructed systematically, whose small-size limit reproduces the boundary condition of Fulop-Tsutsui vertex, (1). The scheme [10] works as follows.

(i) Assemble the edges of n half lines which we assign the numbers $j = 1, 2, ..., n$, and connect them in pairs (i, j) by internal lines of length d/r_{ij} except when $r_{ij} = 0$, for which case, the pairs are left unconnected. Apply vector potential A_{ij} on the line (i, j) to produce extra phase shift χ_{ij} between the edges when its value is nonzero. Place δ potential of strength v_i at each edge i.

(ii) The length ratio r_{ij} and the phase shift χ_{ij} are determined from the non-diagonal elements of the matrix Q defined by

$$Q = \begin{pmatrix} T \\ I^{(n-m)} \end{pmatrix} \begin{pmatrix} -T^\dagger & I^{(m)} \end{pmatrix} = \begin{pmatrix} -TT^\dagger & T \\ -T^\dagger & I^{(m)} \end{pmatrix}, \tag{16}$$

by the realation $r_{ij}e^{i\chi_{ij}} = Q_{ij}$ $(i \ne j)$. This means that we have $r_{ij}e^{i\chi_{ij}} = t_{ij}$ for $i \le m, j > m$, and $r_{ij}e^{i\chi_{ij}} = \sum_{l>m} t_{il}t_{jl}^*$ for $i, j \le m$. For $i, j > m$, we have $r_{ij} = 0$ and naturally also $\chi_{ij} = 0$.

(iii) The strength v_i is given by the diagonal elements of the matrix V defined by

$$V = \frac{1}{d}(2I^{(n)} - J^{(n)})R, \tag{17}$$

where R is the matrix whose elements are made from absolute values of matrix elements of Q, *i.e.* $R = \{r_{ij}\} = \{|Q_{ij}|\}$, The matrix $J^{(n)}$ is of size $n \times n$ with all elements given by 1. This means that we have $v_i = \frac{1}{d}(1 - \sum_{l \le m} r_{li})$ for $i > m$, and $v_i = \frac{1}{d}(\sum_{l>m}[r_{il}^2 - r_{il}] - \sum_{l(\ne i) \le m} r_{il})$ for $i \le m$. These fine tunings of length and strength are necessary to counter the generic opaqueness brought in with every addition of vertices and lines into a graph.

The wave function $\phi(x) = \phi_{i,j}(x)$ on any internal line (i, j), we have the relation

$$\begin{pmatrix} \phi'(0) \\ e^{i\chi}\phi'(\frac{d}{r}) \end{pmatrix} = -\frac{r}{d} \begin{pmatrix} F(\frac{d}{r}) & -G(\frac{d}{r}) \\ G(\frac{d}{r}) & -F(\frac{d}{r}) \end{pmatrix} \begin{pmatrix} \phi(0) \\ e^{i\chi}\phi(\frac{d}{r}) \end{pmatrix}, \tag{18}$$

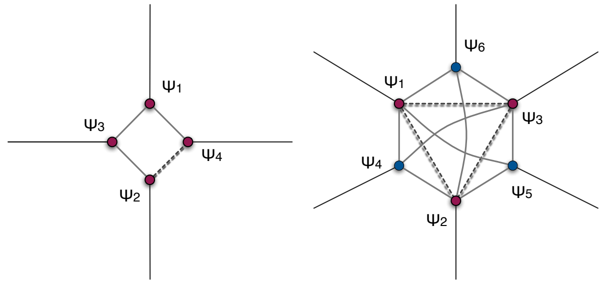

Fig. 1. Finite approximation to the reflectionless Fulop-Tsutsui vertices corresponding to (22) (left) and (28) (right) constructed according to (16)-(17). Dotted line indicates the existence of non-zero phase shift χ_{ij}.

with $F(x) = x \cot x$ and $G(x) = x \operatorname{cosec} x$. Combining (18) with the condition at the i-th endpoint,

$$\psi_i'(0) + \sum_{j \neq i} \phi_{ij}'(0) = v_i \psi_i(0) \tag{19}$$

where we have the δ-potential of strength v_i, we obtain the relations between the boundary values $\psi_i = \psi_i(0)$ and $\psi_i' = \psi_i'(0)$ in the form

$$d\psi_i' = \left(v_i d + \sum_{l \neq i} r_{il} F_{il} \right) \psi_i - \sum_{l \neq i} e^{\mathrm{i}\chi_{ij}} r_{il} G_{il} \psi_l, \tag{20}$$

where the obvious notations $F_{ij} = \frac{d}{r_{il}} \cot \frac{d}{r_{il}}$ and $G_{ij} = \frac{d}{r_{il}} \operatorname{cosec} \frac{d}{r_{il}}$ are adopted. Note that the equation (20) is exact and does not involve any approximation. In the short range limit $d \to 0$, we have $F_{ij} = 1 + O(d^2)$ and $G_{ij} = 1 + O(d^2)$. We can then show, with a straightforward calculation in the manner of [8], that the limit $d \to 0$ gives the desired connection condition for Fulop-Tsutsui vertex (1).

IV. EXAMPLES

With the solution of the inverse scattering fully formulated, it is now possible to find a Fulop-Tsutsui vertex from a given scattering matrix with specific requirement. Our previous results detailed in [10] showing the reconstruction of "Free-like" scattering is one such example, and could have been achieved easier with current method. We now ask whether there is fully reflectionless graph whose scattering matrix has only zeros for its diagonal elements, $\mathcal{S}_{ii} = 0$. Vertices yielding such scattering matrix is known to be useful in developing semiclasical theory of quantum spectra [11]. If we limit ourselves to real $\mathcal{S}$, it becomes symmetric matrix with $\mathcal{S}_{ij} = \mathcal{S}_{ji}$.

We note a useful relation concerning the trace of the scattering matrix. Taking the trace of (6) and utilizing $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we have

$$\operatorname{tr} \mathcal{S} = \operatorname{tr} Z_m = 2m - n. \tag{21}$$

Since $\mathcal{S}$ for reflectionless scattering is traceless, we can have such scattering only for $n = 2m$.

Our first example is with $n = 4$ whose $\mathcal{S}$ is given by

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & a & \sqrt{1-a^2} \\ 0 & 0 & \sqrt{1-a^2} & -a \\ a & \sqrt{1-a^2} & 0 & 0 \\ \sqrt{1-a^2} & -a & 0 & 0 \end{pmatrix}, \tag{22}$$

and the and corresponding T, by

$$T = \begin{pmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & -a \end{pmatrix}. \tag{23}$$

The finite approximation is characterized by

$$r_{12} = r_{34} = 0,\ r_{13} = r_{24} = a,\ r_{23} = r_{14} = \sqrt{1-a^2},$$

$$e^{\mathrm{i}\chi_{24}} = -1,\ e^{\mathrm{i}\chi_{ij}} = 1 \text{ all others,}$$
$$v_1 = v_2 = v_3 = v_4 = \frac{1 - a - \sqrt{1-a^2}}{d}, \tag{24}$$

The finite graph approximation is schematically illustrated in the left side of Figure 1.

We next turn to reflectionless scattering with uniform transmission to all other lines. The smallest non-trivial example of such matrix exists for $n = 4$, and given by

$$\mathcal{S} = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -\mathrm{i} & \mathrm{i} \\ 1 & \mathrm{i} & 0 & -\mathrm{i} \\ 1 & -\mathrm{i} & \mathrm{i} & 0 \end{pmatrix}. \tag{25}$$

The corresponding T is given by

$$T = \begin{pmatrix} \omega & \omega^{-1} \\ \omega^{-4} & \omega^4 \end{pmatrix}. \tag{26}$$

with $\omega = e^{\mathrm{i}\frac{\pi}{6}}$. Our finite approximation is specified by following numbers.

$$r_{13} = r_{14} = r_{23} = r =_{24}= 1,\ r_{12} = r_{34} = 0,$$
$$e^{\mathrm{i}\chi_{13}} = e^{\mathrm{i}\frac{\pi}{6}}, e^{\mathrm{i}\chi_{14}} = e^{-\mathrm{i}\frac{\pi}{6}}, e^{\mathrm{i}\chi_{23}} = e^{-4\mathrm{i}\frac{\pi}{6}}, e^{\mathrm{i}\chi_{24}} = e^{4\mathrm{i}\frac{\pi}{6}},$$
$$v_1 = v_2 = v_3 = v_4 = -\frac{1}{d}, \tag{27}$$

The finite graph approximation is schematically illustrated in the right side of Figure 1.

If we limit ourselves to real scattering matrix, such matrix, called *symmetric conference matrix*, is known to exist for $n = 6, 10, 14, 18, 26, 30, 38, \ldots$. We look at the example of $n = 6$ whose $\mathcal{S}$ is given by

$$\mathcal{S} = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 & -1 & -1 & -1 & 1 & 1 \\ -1 & 0 & -1 & 1 & -1 & 1 \\ -1 & -1 & 0 & 1 & 1 & -1 \\ -1 & 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 1 & 1 & 0 \end{pmatrix}. \tag{28}$$

The corresponding T is given by

$$T = \begin{pmatrix} 1 & 1+\gamma & 1+\gamma \\ 1+\gamma & 1 & 1+\gamma \\ 1+\gamma & 1+\gamma & 1 \end{pmatrix}. \tag{29}$$

where $\gamma = (\sqrt{5}-1)/2$ is the golden mean. Our finite approximation is specified by following numbers.

$$\begin{aligned} r_{12} = r_{23} = r_{13} = 4+3\gamma,\ r_{14} = r_{25} = r_{36} = 1, \\ r_{15} = r_{16} = r_{26} = r_{24} = r_{31} = r_{32} = 1+\gamma, \\ r_{45} = r_{46} = r_{56} = 0, \\ e^{\mathrm{i}\chi_{12}} = e^{\mathrm{i}\chi_{23}} = e^{\mathrm{i}\chi_{13}} = -1,\ e^{\mathrm{i}\chi_{ij}} = 1 \text{ all others}, \\ v_1 = v_2 = v_3 = -6\frac{\gamma+1}{d},\ v_4 = v_5 = v_6 = -2\frac{\gamma+1}{d}. \end{aligned} \tag{30}$$

The finite graph approximation is schematically illustrated in the right side of Figure 1.

Our next example is the reflectionless equitransmitting graph with $n = 10$, that corresponds to the $\mathcal{S}$ matrix given by $n = 10$ conference matrix

$$\mathcal{S} = \frac{1}{3}\begin{pmatrix} 0 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 0 & -1 & 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix} \tag{31}$$

The trace of $\mathcal{S}$ is zero again, and we have $m = \frac{n}{2} = 5$. The matrix T specifying the vertex is given by

$$T = \begin{pmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{pmatrix}, \tag{32}$$

where $\sigma = \sqrt{2}-1$ is the silver mean. Our finite approximation is specified by following numbers for verteces;

$$\begin{aligned} r_{12} = r_{23} = r_{34} = r_{45} = r_{15} = 1, \\ r_{16} = r_{27} = r_{38} = r_{49} = r_{5a} = 1, \\ r_{18} = r_{29} = r_{3a} = r_{46} = r_{57} = 1, \\ r_{19} = r_{2a} = r_{36} = r_{47} = r_{58} = 1, \\ r_{13} = r_{14} = r_{24} = r_{25} = r_{35} = 2, \\ r_{17} = r_{28} = r_{39} = r_{4a} = r_{56} = 0, \\ r_{1a} = r_{26} = r_{37} = r_{48} = r_{59} = 0, \\ r_{67} = r_{78} = r_{89} = r_{9a} = r_{6a} = 0, \\ r_{68} = r_{79} = r_{8a} = r_{69} = r_{7a} = 0, \\ e^{\mathrm{i}\chi_{12}} = e^{\mathrm{i}\chi_{23}} = e^{\mathrm{i}\chi_{34}} = e^{\mathrm{i}\chi_{45}} = e^{\mathrm{i}\chi_{15}} = -1 \\ e^{\mathrm{i}\chi_{16}} = e^{\mathrm{i}\chi_{27}} = e^{\mathrm{i}\chi_{38}} = e^{\mathrm{i}\chi_{49}} = e^{\mathrm{i}\chi_{5a}} = -1 \\ e^{\mathrm{i}\chi_{ij}} = 1 \text{ all others}, \\ v_1 = v_2 = v_3 = v_4 = v_5 = -\frac{6}{d}, \\ v_6 = v_7 = v_8 = v_9 = v_a = -\frac{2}{d}. \end{aligned} \tag{33}$$

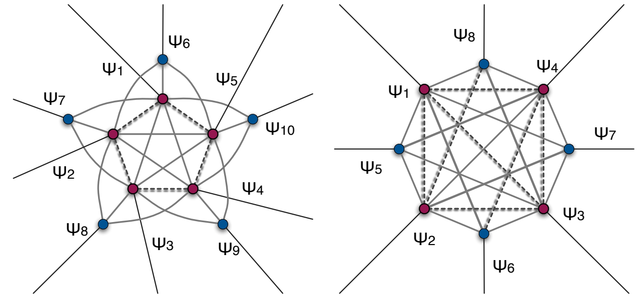

Fig. 2. Finite approximation to the equal-scattering Fulop-Tsutsui vertex corresponding to $n = 10$ conference matrix, (31) (left) and $n = 8$ Hadamard matrix, (34) (right) constructed according to to (16)-(17).

Here, a in subscript stands for the index for 10th edge. The finite graph approximation for this case is schematically illustrated in the left side of Figure 2.

The last example is the equal-scattering graph, in which in the scattering is uniform in all lines including the line of incoming particle. Such matrix, called *symmetric Hadamard matrix*, is known to exist for $n = 2^k$, $k = 0, 1, \ldots$. An example of such $\mathcal{S}$ for $n = 8$ is given by

$$\mathcal{S} = \frac{1}{\sqrt{8}}\begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \end{pmatrix}. \tag{34}$$

The trace of $\mathcal{S}$ is again zero, and we have $m = \frac{n}{2} = 4$. The matrix T specifying the Fulop-Tsutsui the vertex is given by

$$T = \frac{1}{\sigma+1}\begin{pmatrix} \sigma & 1 & 1 & 1 \\ 1 & \sigma & 1 & 1 \\ 1 & 1 & \sigma & 1 \\ 1 & 1 & 1 & \sigma \end{pmatrix}. \tag{35}$$

where $\sigma = \sqrt{2}-1$ is the silver mean. Our finite approximation is specified by following numbers for verteces;

$$\begin{aligned} r_{12} = r_{13} = r_{14} = r_{23} = r_{24} = r_{34} = 1+\sigma, \\ r_{15} = r_{26} = r_{37} = r_{48} = \frac{\sigma}{1+\sigma}, \\ r_{16} = r_{17} = r_{18} = r_{27} = r_{28} = r_{38} = \frac{1}{1+\sigma}, \\ r_{25} = r_{35} = r_{36} = r_{45} = r_{46} = r_{47} = \frac{1}{1+\sigma}, \\ r_{56} = r_{57} = r_{58} = r_{67} = r_{68} = r_{78} = 0, \\ e^{\mathrm{i}\chi_{12}} = e^{\mathrm{i}\chi_{13}} = e^{\mathrm{i}\chi_{14}} = e^{\mathrm{i}\chi_{23}} \\ = e^{\mathrm{i}\phi_{24}} = e^{\mathrm{i}\chi_{34}} = -1,\ e^{\mathrm{i}\chi_{ij}} = 1 \text{ all others}, \\ v_1 = v_2 = v_3 = v_4 = -\frac{5\sigma-3}{d}, \\ v_5 = v_6 = v_7 = v_8 = -\frac{\sigma+1}{d}. \end{aligned} \tag{36}$$

The finite graph approximation is schematically illustrated in the right side of Figure 2.

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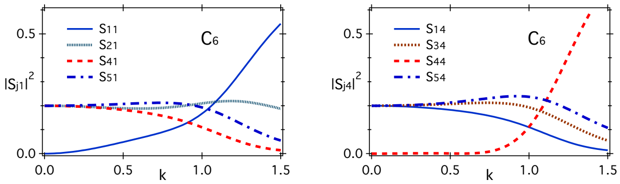

Fig. 3. Scattering probabilities as functions of incoming momentum k (in the unit of $1/d$) of finite quantum graph approximating the equal-transmitting reflectionless vertex with $n = 6$ edges represented in Figure 1, right.

V. Convergence of finite approximations

Finally, we take a look at the convergence of the finite size graph approximation by numerical calculations. In Figure 3, we display the scattering matrix of the finite graph that is constructed to approximate equal-scattering reflectionless matrix, (28). These are calculated directly from (20). The value of the wave length k is in the unit of $1/d$. The convergence can be seen as quite good below $kd \lesssim 0.2$. Numerical analysis of other examples of different graphs give essentially the same conclusion that the construction does represent physical realization of singular Fulop-Tsutsui vertex.

VI. Conclusion and prospects

Now that the problem of finding desired property of Flulop-Tsutsui graph is turned into mathematical one on Hermitian unitary matrix, the search of system with $\mathcal{S}$ having other interesting specifications should follow. various questions would arise along the line, such as whether it is always $\mathrm{tr} S = 0$ for systems with "exchange symmetric" $|Sij|$. Generalization to complex $\mathcal{S}$ is also an interesting problem [11]. Other open questions include the generalization to non-Fulop-Tsutsui connection which yields general unitary $\mathcal{S}$ not limited to Hermitian ones. The study of the bound state spectra is one thing we have completely neglected in this work. Application to non-quantum waves, including electro-magnetic wave and water wave should be another interesting subject.

In this work, through the finite construction of star graph with no internal lines, we have actually studied the low energy properties of graphs with internal lines all of whose edges are connected to external lines, which we might term *depth-one* graphs. The examination of *depth-two* graphs and beyond seems to be the natural future direction. Our result showing the full solution to the inverse scattering problem is, in a sense, a partial fulfillment of the hope that quantum graph somehow could be a solvable model and useful design tool at the same time.

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