Theoretical Study of Micro Flow Instability by Orr-Sommerfeld Equation in Micro Canal

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***Abstract*— Due to interfacial effects that concerns micro channel jets, Plateau-Rayleigh is a well-known instability. In micro fluidics context, the gravity is negligible and surface tension phenomena are predominant. Over the past decade, there has been extensive research into the design of microfluidic systems for chemical analysis. All previous works provided an overview of instabilities that lead to a rich variety of different flow regimes that can be obtained in a micro channel. We survey advancement over ten years in the development of micro scale devices for instability gaseous micro flow. A parametric instability study was systematically conducted with varying system pressure, heat flux, and channel size with and without inlet restrictor. This paper describes various works for micro channel instability flow in gaseous micro fluidic devices. The main objective of this work is the mathematical resolution of the Orr-Sommerfeld equation, and then used this solution to create perturbations in the flow by varying the pressure. The instability of physics is explored using previous theoretical and numerical analyses, as well as experimental observations. The difficulty of the analytical resolution of the Orr-Sommerfeld equation for a velocity profiles for the perturbation has always been a problem; that is why we are going to try to get it numerically.**

Keywords-Gaseous instability; micro canal; microfluidics simulation; experimentation; gaseous flow.

I. INTRODUCTION

Microfluidics is the science and technology of the manipulation of fluids in small channels. The development of the microfluidics science has been seen in the mid of the 70th. The current micro fabrication technology has allowed its rapid expansion from the middle of the 80th [1]. The interest in this science has led to a major development, particularly in the field of chemistry and biology. This growing interest is reflected in several publications.

Drazin and Reid [2] define hydrodynamic instability as a branch of hydrodynamics, which is concerned with "when and how laminar flows break down, their subsequent development and their eventual transition to turbulence." Fluid instabilities may be broadly divided into two classes: convective instability and dynamic instability.

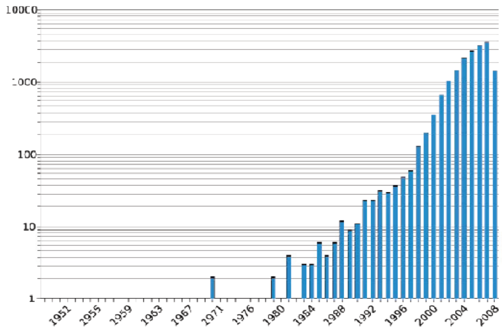

Figure 1. Growth in the number of publications on microfluidics in recent decades [1].

While the convective instabilities can be understood rather easily in terms of forces acting on displaced parcels of fluid, the dynamic one are more varied and more challenging to understand.

When thin cylinders of liquid tend to break up into droplets in the presence of large surface tension; this phenomenon is known as Rayleigh-Plateau instability [2], and it is caused by a positive feedback loop between the decreased cylinder radius and increased tension forces.

As a result, liquid jets and spindles along the boundary of a liquid sheet will reliably break apart as soon as they become thin enough for the surface tension forces to dominate the inertia forces [3] [4]. For applications, we can use coaxial cylinders to create jets, or to generate droplets in order to obtain some micromixers [5]. The classical Rayleigh-Plateau instability is described as a phenomenon in which the velocity inside the jet is constant; the same kind of analysis is done in a more general setting using a long wave approximation [6]. The numerical linear instability is done in the context of a shear flow [7]. K.C Sahu, et al., use the same strategy of Colin and Tancogne [8] in order to study the stability of a coflow composed by a newtonian fluid and a non-newtonian one. The spatial responses of the film to inlet controlled periodic perturbations which have been characterized when the Rayleigh-Plateau is relevant [9].

The emphasis has been put on the linear and nonlinear development of the primary wave train. S. Shabahang et al. [10], present the observation of the Rayleigh-Plateau capillary instability in a multi-material fiber at the core cladding interface.

The Plateau–Rayleigh instability, often just called the Rayleigh instability that is responsible for the phenomenon of the break-up of a jet. Fonade [11] gave some principles of such study focusing primarily on the flow type "jets". The method that is used is similar to the harmonic analysis of a system: in a given mean flow, we add a perturbation which is characterized by velocity and pressure. These characteristics are sinusoidal functions.

The frequency is kept as a parameter; we study how the flow is modified by the perturbation. If the system is nonlinear the answer will depend on the amplitude of the perturbation; therefore, we take low amplitude, which permits linearizing equations.

We consider a parallel of two dimensional flow which defined by $\left[\overrightarrow{U}(y), \overline{p}(x, y)\right]$ in which we superpose a perturbation which defined by $\left[\overrightarrow{U}'(x, y), \overline{p}'(x, y)\right]$.

It is supposed that the mean flow $\left[\overrightarrow{U}, \overline{p}\right]$ and the global flow $\left[(\overrightarrow{U} + \overrightarrow{U}'), (\overline{p} + \overline{p}')\right]$ separately verify the continuity equation and Navier-Stokes equations.

A stream function of the perturbation which is periodic with respect to time, verifies the continuity equation and Navier-Stokes equations. This function is written with the following complex notation:

$$\psi(x, y, t) = \phi(y) e^{i\alpha_1 (x - c_1 t_1)} \quad (1)$$

$\phi(y) = \phi_r + i\phi_i$: The amplitude functions of the perturbation.

α_1 and c_1: constants that are generally complex [Only the real part of the stream function has a physical sense].

When we replace adimensional quantities in the relation (1):

$$\psi = \psi . U_0 L$$
$$x = X . L$$
$$c_1 = c . U_0 = (c_r + i c_i) U_0$$
$$\phi = \phi . U_0 L$$
$$\alpha_1 = \alpha / L = \alpha_r + i\alpha_i \qquad t_1 = tL / U_0$$

The relation (1) becomes:

$$\psi = \phi . e^{[-\alpha_i x + (\alpha_i c_r + \alpha_r c_i) t_1] + i[\alpha_r x - (\alpha_r c_r + \alpha_i c_i) t_1]} \quad (2)$$

The relation (1) is used to calculate the components (u ', v') of the perturbation velocity. Medium flows and global verify the Navier-Stokes equations in which the pressure between these two equations is eliminated, to obtain the equation that is known as "Orr-Sommerfeld" [12].

$$\left(\frac{U}{U_0} - c\right)\left(\frac{d^2\phi}{dY^2} - \alpha^2\phi\right) - \phi \frac{d^2(U/U_0)}{dY^2} = -\frac{i}{\alpha R}\left[\frac{d^4\phi}{dY^4} - 2\alpha^2 \frac{d^2\phi}{dY^2} + \alpha^4\phi\right] \quad (3)$$

R : the Reynolds number $R = \frac{U_0 L}{\nu}$; U : mean flow.

In this equation we must add the boundary conditions that are written to a jet:

- v' = 0 whether $\phi = d\emptyset/dy = 0$ for $Y \rightarrow \pm\infty$
- v' = 0 whether $\phi = 0$ for Y = 0 when we consider a change of a symmetrical perturbation relative to the axis of the jet.
- u = 0 whether $d\emptyset/dy = 0$

(4)

For Y = 0, when we consider an antisymmetric perturbation.

The stability study is to find the eigenvalues of the constants α and c; such as, the solution $\emptyset$ of equation (3) which verifies the boundary conditions.

In addition to the mean flow *U*, and the Reynolds number *R*, we have three parameters $(\alpha_r = \alpha, c_i, c_r)$ in equation (4). $\propto$ is undetermined parameter $(\alpha = 2\pi/\lambda)$ in which λ is the wavelength of the perturbation.

The association between the equation (2) and the boundary conditions (4) provides a solution $\emptyset(y)$ and a pair of values c_r, c_i that are dependent on the Reynolds number *R* and $\propto$. The perturbation will be amplified when $c_i > 0$ and it will be amortized when $c_i < 0$. The perturbation will be neutral or indifferent when $c_i = 0$.

The condition $c_i = 0$ provides the neutral stability curve $f(\alpha, R) = 0$ that can be represented in the plan(α, R).

II. A METHOD OF OBTAINING AN EXACT SOLUTION

We develop a method to obtain an exact solution of Orr-Sommerfeld equation by reducing, the resolution of this differential equation solving a Volterra integral equation.

We observe that:

$$(U - c)(\varphi'' + \alpha^2\varphi) - U''\varphi = -\frac{i}{\alpha R}(\varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi) \quad (5)$$

$$(\varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi) = (\varphi'' + \alpha^2\varphi)'' - \alpha^2(\varphi'' + \alpha^2\varphi)$$

We introduce the function *f* by the differential equation:

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$\varphi'' - \alpha^2 \varphi = f$ (Change of variable)

$r^2 - \alpha^2 = 0 \quad ; \quad r_1 = \alpha \ , r_2 = -\alpha$

The solution of the differential equation

$$\varphi'' - \alpha^2 \varphi = 0$$

is:

$$\varphi(p) = C_1(p) e^{\alpha p} + C_2(p) e^{-\alpha p}$$

where p is the parameter of integration

$$\begin{cases} C'_1(p) e^{\alpha p} + C'_2(p) e^{-\alpha p} = 0 \\ \alpha C'_1(p) \left(e^{\alpha p}\right)' - \alpha C'_2(p) \left(e^{-\alpha p}\right)' = f(p) \end{cases}$$

$$\Rightarrow \varphi(y) = \frac{1}{\alpha} \int_0^y f(p) \sinh \alpha (y-p) dp + c_1^* e^{\alpha y} - c_2^* e^{-\alpha y}$$

$$\Rightarrow \varphi(y) = \frac{1}{\alpha} \int_0^y f(p) \left(\frac{e^{\alpha(y-p)} - e^{-\alpha(y-p)}}{2} \right) dp + C_1 \cosh \alpha y + C_2 \sinh \alpha y$$

$$\Rightarrow \varphi(y) = \left(\frac{1}{\alpha} \int_0^y \frac{1}{2} f(p) e^{-\alpha p} dp \right) e^{\alpha y} - \left(\frac{1}{\alpha} \int_0^y \frac{1}{2} f(p) e^{\alpha p} dp \right) e^{-\alpha y} + \left(c_1^* + c_2^*\right) \cosh \alpha y + \left(c_1^* - c_2^*\right) \sinh \alpha y$$

$$\Rightarrow \varphi(y) = \frac{1}{\alpha} \int_0^y f(p) \sinh \alpha (y-p) dp + C_1 \cosh \alpha y + C_2 \sinh \alpha y \qquad (6)$$

Using the boundary condition in (6) $\varphi(0) = \varphi'(0) = 0$

$$\varphi'(y) = \frac{1}{\alpha} f(p) \sinh \alpha (y-y) + \alpha C_1 e^{\alpha y} + \alpha C_2 e^{-\alpha y}$$

$$\begin{cases} \varphi(0) = C_1 - C_2 = 0 \\ \varphi'(0) = \alpha C_1 + \alpha C_2 = 0 \end{cases}$$

$$\begin{cases} C_1 - C_2 = 0 \\ C_1 + C_2 = 0 \end{cases} \Rightarrow C_1 = C_2 = 0$$

Then the equation (3):

$$(1) \Rightarrow (U-c) f - U'' \frac{1}{\alpha} \int f(p) \sinh \alpha (y-p) dp = -\frac{i}{\alpha \mathrm{Re}} (f'' - \alpha^2 f) \qquad (7)$$

A new function ϕ is introduced by the equation $f'' - \alpha^2 f = \phi$ (change of variable)

With the same steps we obtain the following solution:

$$f(y) = \frac{1}{\alpha} \int_0^y \phi(p) \sinh \alpha (y-p) dp + A \cosh \alpha y + B \sinh \alpha y \qquad (8)$$

If we replace f in the equation (7) we obtain:

$$[(U-c) \frac{1}{\propto} \emptyset(p) sinh \propto (y-p) dp + Acosh \propto y -$$

$$U'' \frac{1}{\propto} \int_0^y f(p) sinh \propto (y-p) dp = -\frac{i}{\propto Re} \emptyset(y)] . i \propto Re$$

$$\Rightarrow \phi(y) - i \,\mathrm{Re}(U-c) \int_0^y \phi(p) \sinh \alpha (y-p) dp +$$

$$i \,\mathrm{Re}\, U'' \int_0^y f(p) \sinh \alpha (y-p) dp = i \alpha \,\mathrm{Re}\left(A \cosh \alpha y + B \sinh \alpha y\right)$$

If we replace $f(p)$ by the expression (8), we obtain:

$$\phi(\mathrm{y}) - \mathrm{iRe} \int_0^y \left(\frac{U''}{\alpha \phi(p)} \int \phi(s) \sinh \alpha (y-s) ds - (U-c) \right) .$$
$$. \sinh \alpha (y-p) \phi(p) dp = i (i \alpha \,\mathrm{Re}\left(A \cosh \alpha y + B \sinh \alpha y\right) -$$
$$- \alpha \,\mathrm{Re} \int_0^y \left(A \cosh \alpha y + B \sinh \alpha y\right) \sinh \alpha (y-p) dp) \qquad (9)$$

and if we pose :

$$i \alpha \mathrm{Re}\left(A \cosh \alpha y + B \sinh \alpha y\right) - \alpha \mathrm{Re} \int_0^y \left(A \cosh \alpha y + B \sinh \alpha y\right)$$
$$\sinh \alpha (y-p) dp = F(y)$$

We obtained:

$$\phi(y) - i \mathrm{Re} \int_0^y \left(\frac{U''}{\alpha \phi(p)} \int_0^p \phi(s) \sinh \alpha (y-s) ds - (U-c) \right) \sinh \alpha (y-p) \phi(p) dp = i F(y) \qquad (10)$$

Posing:

$$K(y,p) = \int_0^y \left(\frac{U''}{\alpha \phi(p)} \int_0^p \phi(s) \sinh \alpha (y-s) ds - (U-c) \right) \sinh \alpha (y-p)$$

We obtain:

$$\phi(y) + i \,\mathrm{Re} \int_0^y K(y,p) \phi(p) dp = -iF(y) \qquad (11)$$

As a result, by the complex integral equation of the Volterra type of the second species, we can obtain the solution of the differential equation φ(y) of the Orr-Sommerfeld equation by a simple quadrature.

$$\varphi = \frac{1}{2\alpha^2} \int_0^y [(y-p) cosh\alpha(y-p) - \frac{1}{\alpha} sinh\alpha(y-p)] x$$
$$\emptyset(p) dp + \frac{1}{2\alpha} y sinh \propto y +$$
$$+ \frac{B}{2\alpha} \left(y cosh \propto y - \frac{1}{\alpha} sinh \propto y \right) \qquad (12)$$

III. SIMULATION OF MICROCHANEL T JONCTION GEOMETRIES

The boundary condition: P_{inlet1}= 2 bars, P_{inlet2}= 2.3, 2.5, 3 bars and P_{outlet} = 1 bar.

The descriptions of the simulated geometry with its dimensions are shown in Figure 2.

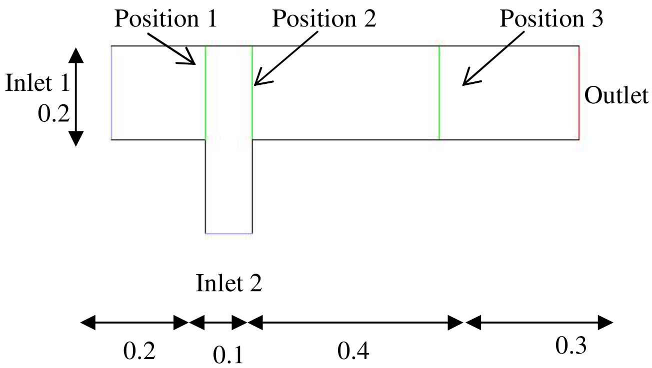


Figure 2. Description of schematic geometry (mm).

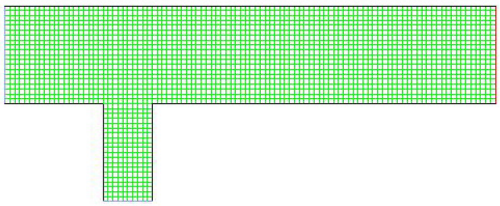
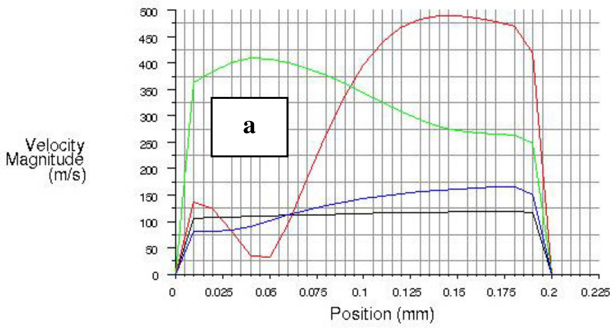


Figure 3. Meshing geometry of simulation.

IV INTERPRETATION OF RESULTS AND PERSPECTIVES

The following results represent the velocity profiles in different positions 1, 2 and 3 for different pressures a, b and c (Figure 4).



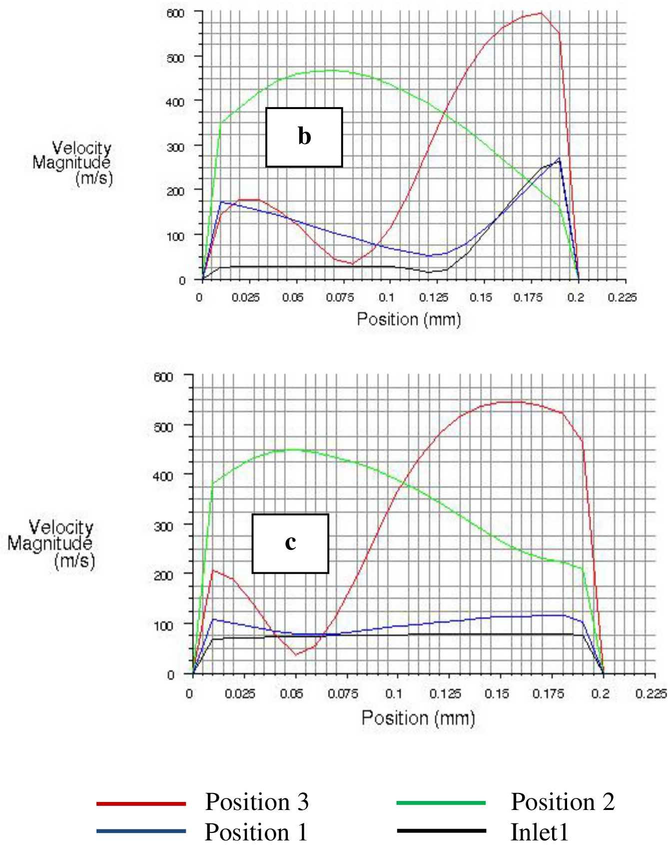


Figure 4. The velocity (m/s).
a: P_{Inlet2}=2.3 bars, b: P_{Inlet2}=2.5 bars, c: P_{Inlet2}=3 bars

With the variation of the entry pressure (P Inlet2), the velocities to the wall equals: zero (Figure 4: a, b, c).

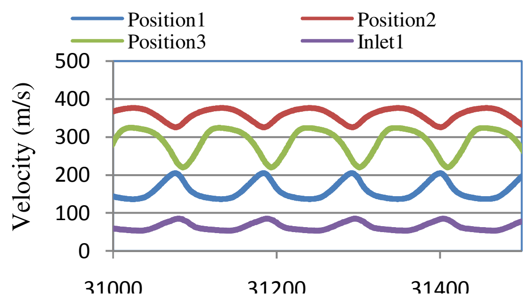


Figure 5: Velocity (m/s) in different position (PInlet2 = 3 bars)

In table 1, the disruption caused a variation (increase) of the velocity is shown.

TABLE1. THE MAXIMAL TRANSVERSAL VELOCITY (m/s) IN DIFFERENT POSITIONS FOR DIFFERENT PRESSURES INLET2.

Pressure Inlet2 (bars)		2.3	2.5	3
maximal transversal Velocity (m/s)	Inlet1	113,69	77,72	61.024105
	position1	140,36	118,02	136.13811
	position2	329,42	342,59	399.43102
	position3	309,67	325,83	382.96531

We note that the transversal velocity at a position 2 is always maximum comparing at others positions with any pressure inlet 2 and it increases when the pressure inlet 2 increases.

V. CONCLUSION

Microfluidics deals fluid flow, single or multiphase systems in very small dimensions in the micrometer range. In this paper, we present a study of jet created by a perturbation as different pressures.

Our research team is in the process of developing a new line of research in our laboratory, the study of the hydrodynamic instability of gas flow.

The equation of the perturbation is known by the Orr Sommerfeld equation. The difficulty of the analytical resolution of this equation is always a problem for obtained a velocity profiles for the perturbation we try to obtain it numerically.

The first result consist a manipulation of all step of mathematical resolution of Sommerfeld equation. Really the difficulty is located on resolution of Voletra equation.

The numerical results were presented. The velocity profile at different position of geometry indicates the sensibility of pressures. We conclude the perturbation of principal flow is directly influenced by the size of elementary channel and pressures.

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