Quantum States in Bivalent Logic

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***Abstract*—Bivalent or two-valued logic is presently the foundation of logic in mathematics and computer science, and a cornerstone of software development. To address a number of classical logical paradoxes, such as Russell's, multi-valued logic, such as balanced ternary logic has shown to be useful. Current methods lead however to information loss. Thus, to theoretically improve the robustness of bivalent logic, this paper proposes the use of quantum states, followed by an example, where the proposed method is shown to be successful in the solution of a problem that is not directly solvable using contemporary methods.**

Keywords-bivalent logic; propositional logic; quantum state; Russell's paradox; ternary logic

I. INTRODUCTION

On the topics of fundamentals in software development and information modeling, the logical systems today are either typically based on static bivalent logical values, such as true and false, or static fuzzy logic values. This paper does not address quantum logic [1], or quantum computing, where the laws of logic are expanded for application in quantum mechanics, but on the contrary, the application of results from quantum mechanics to classical bivalent logic. In bivalent logic, contradictions, such as Russell's paradox [6], or Epimenides paradox [5] (which may be expressed as "this statement is false"), cause an infinite loop of alternating values such as:

$$\text{true} \rightarrow \text{false} \rightarrow \text{true} \rightarrow \text{false} \ldots \tag{1}$$

Russell's paradox hypothesizes the existence of a set that does not contain itself. Epimenides paradox may be expressed as:

$$x \equiv \neg x \tag{2}$$

In the same way, that resonance may cause instability in control theory, such contradictions may cause instability in machine reasoning, e.g., machine interpretation of propositional logic.

To make a comparison with the field of robotics, it takes in general less effort to program an industrial robot that works in a highly structured environment, than a robot working in an unstructured one, where for instance the precise position, orientation, or the geometry of a workpiece are not always known in advance. Similarly, if we wish to develop machines that are able to handle and solve logical problems in the real world, we need to strive towards the incorporation of a higher level of flexibility in machine information processing, e.g., a higher level of tolerance towards contradictions.

Presently, computer simulations of logical statements (using modern computer languages such as C++ or Java), that include a paradox such as Epimenides or Russell's, tend to either cause the simulation to yield incorrect results, or the program to fall into an infinite loop, why such problems are presently solved manually. The aim of this paper is, therefore, to propose a new method that makes the evaluation of logical statements, that include a paradox (such as Epimenides paradox), intrinsically solvable to computers.

As a brief overview of this paper, Section II succinctly reiterates the state of the art in paradox-tolerant logic. Section III presents a proposal with the aim to further the methods within this field, and in Section IV, the new proposal is verified by computational experiments. Finally, in Section V, an example is provided on the application of the new method in comparison with current ones.

II. RELATED WORK

In context of paradox-tolerant logic, the design of a versatile system was addressed by Lukasiewicz [3] in 1920, using a balanced ternary (three-valued) logic. In this system:

$$\begin{aligned} 1 &\leftarrow \text{true} \\ 0 &\leftarrow \text{unknown} \\ -1 &\leftarrow \text{false} \end{aligned} \tag{3}$$

With the definition of $\neg x$ as $-x$, (2) is solved by:

$$x = -x \Rightarrow x = 0 \tag{4}$$

In addition to negation (not), see Table I, representing Boolean logic [2], other logical connectives may be introduced as well, such as conjunction (and), disjunction (or), implication ($\rightarrow$), and equivalence ($\leftrightarrow$).

In Lukasiewicz logic, conjunction $(x \wedge y)$ may be defined as $\min(x, y)$, and disjunction $(x \vee y)$ as $\max(x, y)$, which, as shown in Table II, produce reasonable results. The downside of this approach is that by using zero to for instance represent a logical wave, we have effectively lost information regarding the phase of this wave for further analysis down the line.

TABLE I. BOOLEAN LOGIC

x	y	$\neg x$	$x \wedge y$	$x \vee y$	$x \rightarrow y$	$x \leftrightarrow y$
0	0	1	0	0	1	1
0	1	1	0	1	1	0
1	0	0	0	1	0	0
1	1	0	1	1	1	1

In this context, another system of interest is the four-valued "Diamond" logic [4], which structurally has many distinct similarities with the proposal presented in this paper. In "Diamond" logic, in addition to the values true and false, two new values are introduced, called i and j, where by definition $i \equiv \neg i$ and $j \equiv \neg j$. This definition resolves the contradiction in (2), but leads from a perspective, to the generation of a new set of contradictions, such as while $i \vee \neg i$ is expected to be a

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TABLE II. A BALANCED TERNARY LOGIC WITH $\neg x \equiv -x$, $x \wedge y \equiv \min(x, y)$, $x \vee y \equiv \max(x, y)$, $x \rightarrow y \equiv \neg x \wedge y$, AND $x \leftrightarrow y \equiv (x \wedge y) \vee (\neg x \wedge \neg y)$

x	y	$\neg x$	$x \wedge y$	$x \vee y$	$x \rightarrow y$	$x \leftrightarrow y$
−1	−1	1	−1	−1	1	1
−1	1	1	−1	1	1	−1
1	−1	−1	−1	1	−1	−1
1	1	−1	1	1	1	1
0	1	0	0	1	1	0
1	0	−1	0	1	0	0
0	−1	0	−1	0	0	0
−1	0	1	−1	0	1	0
0	0	0	0	0	0	0

tautology, since $i \equiv \neg i$, instead $i \vee \neg i \equiv i$, which is caused by information loss.

III. PROPOSAL

In the new proposal, based on two quantum states, ψ and $\bar{\psi}$, using a 2D Boolean vector:

$$\begin{aligned} \text{false} &= 0 = 00_2 \\ \bar{\psi} &= 1 = 01_2 \\ \psi &= 2 = 10_2 \\ \text{true} &= 3 = 11_2 \end{aligned} \tag{5}$$

$$\text{false} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \bar{\psi} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \text{true} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6}$$

In this system, all logical connectives are expected to operate element-wise on the 2D Boolean vectors. Thus:

$$\bar{\psi} \equiv \neg\psi, \quad \psi \equiv \neg\bar{\psi} \tag{7}$$

Further on, the equation $x \equiv \neg x$ is here regarded as a discrete wave equation, in essence, similar to the time-dependent Schrodinger equation in quantum mechanics [7]:

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r}, t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}, t)\right]\Psi(\mathbf{r}, t) \tag{8}$$

where any wave function that can satisfy this equation is called a "quantum state". A crucial point here is however that while in Lukasiewicz logic, the relation $x \equiv \neg x$ is fully satisfied (and by mere definition in "Diamond" logic), in this new proposal, $x \equiv \neg x$ is only satisfied for ψ and $\bar{\psi}$ by the substitution of x with $\neg x$ on the right hand side of the equation, thereby yielding the solution $x \equiv \neg(\neg x)$. However, in this new approach, the phase of the logical wave is in addition preserved.

IV. RESULTS

According to the results presented in Tables III-IV, the use of quantum states appears to yield a truth table for the selected connectives that preserves the phase of the wave function for further calculations down the line. In this context, it seems however that although the results for $x \leftrightarrow y$ is technically correct (which is equivalent to $x \equiv y$), it would be plausible to define a stronger connective for equivalence as well. In Tables III-IV, "strong" equivalence is denoted as an equivalent sign with four lines instead of three.

TABLE III. FOUR-VALUED LOGIC BASED ON 2D BIVALENT LOGIC, WITH $x \rightarrow y \equiv \neg x \wedge y$, AND $x \leftrightarrow y \equiv (x \wedge y) \vee (\neg x \wedge \neg y)$

x	y	$\neg x$	$x \wedge y$	$x \vee y$	$x \rightarrow y$	$x \leftrightarrow y$	≣
00	00	11	00	00	11	11	11
00	01	11	00	01	11	10	00
00	10	11	00	10	11	01	00
00	11	11	00	11	11	00	00
01	00	10	00	01	10	10	00
01	01	10	01	01	11	11	11
01	10	10	00	11	10	00	00
01	11	10	01	11	11	01	00
10	00	01	00	10	01	01	00
10	01	01	00	11	01	00	00
10	10	01	10	10	11	11	11
10	11	01	10	11	11	10	00
11	00	00	00	11	00	00	00
11	01	00	01	11	01	01	00
11	10	00	10	11	10	10	00
11	11	00	11	11	11	11	11

TABLE IV. SAME AS PREVIOUS TABLE, BUT HERE WITH $0 \leftarrow 00_2$, $\psi \leftarrow 01_2$, $\bar{\psi} \leftarrow 10_2$, AND $1 \leftarrow 11_2$

x	y	$\neg x$	$x \wedge y$	$x \vee y$	$x \rightarrow y$	$x \leftrightarrow y$	≣
0	0	1	0	0	1	1	1
0	ψ	1	0	ψ	1	$\bar{\psi}$	0
0	$\bar{\psi}$	1	0	$\bar{\psi}$	1	ψ	0
0	1	1	0	1	1	0	0
ψ	0	$\bar{\psi}$	0	ψ	$\bar{\psi}$	$\bar{\psi}$	0
ψ	ψ	$\bar{\psi}$	ψ	ψ	1	1	1
ψ	$\bar{\psi}$	$\bar{\psi}$	0	1	$\bar{\psi}$	0	0
ψ	1	$\bar{\psi}$	ψ	1	1	ψ	0
$\bar{\psi}$	0	ψ	0	$\bar{\psi}$	ψ	ψ	0
$\bar{\psi}$	ψ	ψ	0	1	ψ	0	0
$\bar{\psi}$	$\bar{\psi}$	ψ	$\bar{\psi}$	$\bar{\psi}$	1	1	1
$\bar{\psi}$	1	ψ	$\bar{\psi}$	1	1	$\bar{\psi}$	0
1	0	0	0	1	0	0	0
1	ψ	0	ψ	1	ψ	ψ	0
1	$\bar{\psi}$	0	$\bar{\psi}$	1	$\bar{\psi}$	$\bar{\psi}$	0
1	1	0	1	1	1	1	1

V. APPLICATION

As an example regarding the new proposal, we consider a problem that is straightforward to figure out for a human, but presently, relatively hard for a machine to solve without any additional assistance.

Problem. A family consists of two parents and two children, A and B. A child that has received the house key will use it to unlock a door for both children. According to the statements made by the parents, $s_1 - s_3$, where the statements are assumed to be mutually synchronized:

s_1: All statements ($s_1 - s_3$) are false.

s_2: Child A is in possession of the key.

s_3: Child B is not in possession of the key.

The question is hence, are the children able to unlock the door?

Approach 1. Since $s_1 = \neg s_1$ (Epimenides paradox), the use of quantum states, according to the proposal of this paper, yields the solutions: $\{s_2 = \psi, s_3 = \bar{\psi}\}$ and $\{s_2 = \bar{\psi}, s_3 = \psi\}$.

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Since $\bar{\psi} \vee \psi \equiv \psi \vee \bar{\psi} \equiv$ true, both solutions yield the correct conclusion that the children are able to unlock the door.

Approach 2. Using Lukasiewicz logic, since $s_1 = \neg s_1$, the (fuzzy-logical) value of s_1 is equivalent to zero, since according to (4), $x = \neg x \Rightarrow x = 0$. Thus, $s_2 = 0$, and since $s_2 = \neg s_3$, thereby, $s_3 = 0$. Further, since $s_2 \vee s_3 = 0$, this yields that we are not able to establish whether any of the children are in possession of the key, and therefore able to unlock the door. No conclusion can thus be drawn.

Approach 3. Using "Diamond" logic, given $s_1 = \neg s_1$, we are able to either define the solution of s_2 as i or j. However, since in the case s_3 is equal to i (or alternatively j), and since $s_2 = \neg s_3$, but $i = \neg i$ (or alternatively $j = \neg j$), this creates a new set of paradoxes, that yield both correct and incorrect solutions. Thus, no univocal conclusion can be drawn.

VI. CONCLUSION

The results of this paper raise questions on the nature of the fundamental building blocks of logic. The logical wave functions ψ and $\bar{\psi}$, as defined here, cannot be directly derived by the static scalar values true or false, but the opposite holds, since true $= \psi \vee \bar{\psi}$, and false $= \psi \wedge \bar{\psi}$. Further on, while static logical values seem to be the root cause of many contradictions in logic, such as Russell's paradox, as shown here, this issue may instead be addressed using quantum states as the building blocks of mathematical logic.

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