# A Distribution for Service Model

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***Abstract*—In this paper, we developed a flexible service model for the minimum service time called Minimum-Conway-Maxwell-Poisson-exponential distribution, denoted by MINCOMPE distribution, with the service rate dependent on the state of the system including the idle period. This distribution is a new approach where it is possible to look only at the service and capture variations of the system. In addition, this distribution is to model the dependency between the interarrival and service times. The MINCOMPE distribution contains submodels, such as Minimum-geometric-exponential, Minimum-Poisson-exponential and Minimum-Bernoulli-exponential, which express variations of the system. The properties of the proposed distribution are discussed, including formal proof of its probability density function and explicit algebraic formulas for their reliability and moments. The parameter estimation is based on the usual maximum likelihood method. Simulated and real data are shown to illustrate the applicability of the model.**

**Keywords**:**Conway-Maxwell-Poisson distribution; MINCOMPE distribution; minimum service time.**

## I. INTRODUCTION

In this paper, we studied a specific system where the interarrival times are the same as the service times. In this system, there is a dependency between the interarrival and service times where the service is attached to the arrival. Hence, when the service finishes another customer arrives in the system and enters into the service directly. When the number of customers increases, the service becomes faster and the interarrival time decreases. Therefore, it is necessary to have an adjustement mechanism in order to restablish the balance of the system. The possible adjustments are to change the service rate and/or the opening of new service channels.

We proposed a distribution that describe this system which we called Minimum-Conway-Maxwell-Poisson-exponential distribution, denoted by MINCOMPE distribution, with service rate dependening on the state of the system. The MINCOMPE distribution contains various submodels, which can be obtained by varying the pressure parameter, such as, Minimum-geometric-exponential, Minimum-Poisson-exponential and Minimum-Bernoulli-exponential. This submodels capture the oscillations of the system due to the increase of the number of curtomers.

The MINCOMPE distribution was obtained using a compound of two distributions, the Conway-Maxwell-Poisson for the number of customers, denoted by COM-Poisson, and the exponential distribution for the interarrival time. The main goal was to observe the minimum interarrival times when the number of customers in the system is unknown.

It is necessary to consider the following system for the compound; a single server where the service time is exponential distributed and the mean depends on the system state and is given by $\mu_m = m^{\phi}\mu$, where the number of customers is indicated by $m$. The degree to which the service rate is affected by the system state is indicated by $\phi$ and it is called pressure parameter; the arrivals in the system occur at random; the interarrival times are exponentially distributed with mean $\lambda$; the customers are served on a First-Come-First-Served (FCFS).

It is generally believed that, the usual queue model has the service rate independent of the system state however, it is a special case when $\phi = 0$ so that $\mu_m = \mu$ for all $m$. Moreover, when $\phi = 1$ the service rate is directly proportional to the system state and the opening of new service channels. When $\phi$ values are greater than one, the service rate is more proportional to an increase in work.

In other words, the pressure parameter is a defense mechanism when there is a backlog of work. An increase in effort on the part of the server is an obvious source of increase in service rate.

Moreover, ”the Poisson arrivals see Time Averages property, denoted by PASTA, was used, meaning when arrivals are Poisson, the fraction of arrivals who find a process in some state (busy or idle) is equal to the fraction of time the process is in that state”, this property is described in [1].

In the literature, there are few references to be considered compound and state dependent service rate. We would like to mention the modeling studies: Jongbloed and Koole [2] studied a call center as a queueing model with Poisson arrivals having an unknown varying arrival rate. Srikanth and Manjunath [3] analyze queueing models where the joint density of the interarrival time and the service time were described by a mixture of joint densities.

This paper has been organized as follows. In Section A, we presented the MINCOMPE distribution and some of its properties for minimum interarrival time or minimum service time.

In Section B, we derived the expressions for the probability density function, and r-th raw moments of the MINCOMPE distribution.

In Section C, we described the maximum likelihood estimation of the parameters of the model and demonstrated some numerical results with simulation and real data. Finally, Section II contains final remarks.

*A. The Distribution for Minimum Service Time*

The process can be described as follows. Let be $M$ a random variable denoting the number of customers in the system, $m = 0,1,2\ldots$, with COM-Poisson distribution described in [4] and [5], with probability mass function (pmf) expressed as

$$P_m(M=m;\rho,\phi) = \frac{1}{[Z(\rho,\phi)]}\frac{\rho^m}{(m!)^\phi}, m=0,\ldots \tag{1}$$

where $Z(\rho,\phi) = \sum_{j=0}^{\infty} = \frac{\rho^j}{(j!)^\phi}$ is normalizing constant, $\rho$ is traffic intensity with $\rho < 1$ and $\phi \in (-\infty,\infty)$. The stability condition $\rho = \lambda/\mu < 1$ means that the arrival rate $\lambda$ must be less than the service rate $\mu$.

"Note that, the COM-Poisson distribution is undefined when $\rho \geq 1$, $\phi = 0$. Extending $\phi$ to its two extremities, the COM-Poisson distribution in (1) can be seen as a continuous bridge between the geometric ($\phi = 0$, *with* $0 < \rho < 1$), the Poisson ($\phi = 1$) and Bernoulli ($\phi \to \infty$) distributions. This distribution is overdispersed when $\phi \in [0,1)$ and underdispersion when $\phi \in (1,\infty)$" described in [4].

We assumed that the interarrival times and the service times follow the exponential distribution. Let $Y_i$, $i = 1,2,\ldots$ be random variables denoting interarrival times exponentially distributed with mean $\lambda$ and given by

$$f(y_i;\lambda) = \lambda e^{-\lambda y_i}. \tag{2}$$

Most queueing models assume that interarrival times are statistically independent of the service times. However, such an assumption is not always valid. It is also important to take in consideration the possibility of the arrivals of customers being attached to the service as a control of the flow of customers or queue control models. If that is the case, customers will arrive in the system when a services finish. In other words there is no difference between the interarrival time and the service time.

In this paper, we are interested in observing only the minimum interarrival times or minimum service time as this represents how fast the system works and it is given by $Y$

$$Y = min[Y_1,\ldots Y_m], \tag{3}$$

considering that this is a crucial fact in order to stablish customer loyaty.

Considering the dependence between interarrival times and service times, we derived the distribution of the minimum interarrival or minimum service time given by a compound COM-Poisson distribution for the number of customer and exponential distribution for service time. Therefore, if $Y_i$ and $M$ are densities given by (2) and (1) respectively, the minimum service time distribution is given by

$$f_Y(y,\theta) = \frac{\lambda}{Z(\rho,\phi)}\sum_{m=1}^{\infty} m\frac{\rho^m e^{-m\lambda y}}{(m!)^\phi}\ , y>0, \tag{4}$$

where $\theta = (\rho,\lambda,\phi)^T$. In addition, (4) can be rewritten in the form

$$f_Y(y,\theta) = \frac{\lambda Z_1(\rho e^{-\lambda y},\phi)}{Z(\rho,\phi)}E(M_1), \tag{5}$$

where $M_1 \sim COM-Poisson(\rho e^{-\lambda y},\phi)$.

In addition $Z_1(\rho,\phi) = \sum_{j=0}^{\infty} = \frac{(\rho e^{\lambda y})^j}{(j!)^\phi}$ is normalizing constant and $E(M_1) = \rho e^{\lambda y} d\log Z_1(\rho e^{-\lambda y},\phi)/d\rho$.

Therefore, the random variable $Y$ has an MINCOMPE distribution if the cumulative distribution function takes the form

$$F_Y(y;\theta) = 1 - \frac{Z_1(\rho e^{-\lambda y},\phi)}{Z(\rho,\phi)}. \tag{6}$$

We rewritten (4) using the misture of exponential distribution and it is given by

$$f_Y(y,\theta) = \sum_{m=0}^{\infty} v_m f_E(y,m\lambda), \tag{7}$$

where $f_{E_Y}(y,\theta)$ denotes the exponential distribution function with parameter $\lambda$ and the coefficient $v_m$ was represented by COM-Poisson probabilities given by

$$\begin{aligned} v_m &= v_m(\rho,\phi) = P_m(M=m;\rho;\phi) \\ &= \frac{1}{Z(\rho,\phi)}\frac{m\rho^m}{(m!)^\phi}. \end{aligned} \tag{8}$$

where $\sum_{m=0}^{\infty} v_m = 1$. **Therefore, (6) can be rewritten and takes the form**

$$F_Y(y;\theta) = 1 - \sum_{m=0}^{\infty} v_m e^{-m\lambda y}. \tag{9}$$

The moments of the MINCOMPE distribution can be immediately obtained as linear functions of the exponential moments as

$$E(Y^r) = \lambda^{-r}\Gamma(r+1)\sum_{m=1}^{\infty} v_m m^{-r}. \tag{10}$$

In (11), the reliability function is shown

$$W_Y(y,\theta) = \frac{Z_1(\rho e^{-\lambda y},\phi)}{Z(\rho,\phi)}. \tag{11}$$

Thus, reliability function is the probability of no failures in the interval $[0,y]$ or equivalently, the probability to observe the service time after $y$ time.

When the number of the arrival of customers in the system increases consequentely the interarrival times decrease. Due to this fact, there is a continiously pressure on the server to attend the high demand of work. Therefore, the system has an adjustement mechanism in order to restablish the balance of the system. In this case, the possible adjustements are to

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change the service rate and/or open new service channels. These adjustements are captured according to the variations in the pressure parameter and described by corollaries below.

*Corollary 1:* When $\phi = 0$, the MINCOMPE distribution becames the Minimum-geometric-exponential distribution, denoted by MINGE distribution, for the minimum service time. The COM-Poisson is reduced to a geometric distribution and the service rate is independent of the system state. Therefore, the server is not accelerated and it is not stressed with the arrival of the customers. It is not necessary to do an adjustement in the system.

Therefore, the MINGE distribution is given by

$$f_Y(y,\theta) = \frac{\lambda e^{-\lambda y}(1-\rho)}{(1-\rho e^{-\lambda y})^2}. \tag{12}$$

The reliability function is obtained by

$$W_Y(y,\theta) = \frac{(1-\rho)e^{-\lambda y}}{(1-e^{-\lambda y}\rho)}. \tag{13}$$

When $\phi = 0$ in (10) the raw moments of $Y$ is obtained and it is given by

$$E(Y^r) = \lambda^{-r}\Gamma(r+1)(1-\rho)\sum_{m=0}^{\infty}\sum_{k=0}^{m-1}\rho^m(k)^{-r}. \tag{14}$$

*Corollary 2:* When the pressure parameter assumed $\phi = 1$, the MINCOMPE distribution is reduced to the Minimum-Poisson-exponential distribution, denoted by MINPE distribution, for the minimum service time. The COM-Poisson is reduced to a Poisson distribution and the service rate is directly proportional to the system state and the server is accelerated. The adjustements mechanisms in order to restablish the balance of the system are opening of new service channel proportional to the number of customers and increase the service rate.

When $\phi = 1$ in (4) we obtained the MINPE distribution and it is given by

$$f_Y(y,\theta) = \frac{\lambda\rho e^{-\rho-\lambda y+\rho e^{-\lambda y}}}{(1-e^{-\rho})}\ y > 0. \tag{15}$$

The reliability function is given by

$$W_Y(y,\theta) = \frac{e^{\rho e^{-\lambda y}}}{(e^{\rho})}. \tag{16}$$

When $\phi = 1$ in (10) the raw moments of $Y$ is obtained and it is given by

$$E(Y^r) = \lambda^{-r}\Gamma(r+1)e^{-\rho}\sum_{m=0}^{\infty}\rho^m(m!)^{-r}. \tag{17}$$

*Corollary 3:* When $\phi \to \infty$, the MINCOMPE was converted to Minimum-Bernoulli-exponential distribution, denoted by MINBE, for the minimum service time. The COM-Poisson is reduced to a Bernoulli distribution and the service rate is dependent of the system state. The server is accelerated, consequently the service rate increased.

If the random variable $Y$ was defined as (3), replacing $\phi \to \infty$ in (4) and it is given by

$$f_Y(y,\theta) = \frac{\rho\lambda e^{-\lambda y}}{(1+\rho)}. \tag{18}$$

Therefore, the reliability function was presented by

$$W_Y(y,\theta) = \frac{1+\rho e^{-\lambda y}}{(1+\rho)}. \tag{19}$$

The raw moment of the exponential distribution was given by

$$E(Y^r) = \lambda^{-r}\Gamma(r+1)\frac{\rho}{(1+\rho)}. \tag{20}$$

### B. *Maximum Likelihood Estimation*

The maximum likelihood estimation is considered the log-likelihood MINCOMPE distribution in (5) can be written as

$$\begin{aligned}\ell(\theta,y) &= -n\log Z(\rho,\phi)+\sum_{i=1}^{n}\log(Z_1(\rho e^{-\lambda y_i},\phi)) \\ &+ \sum_{i=0}^{n}\log E[M_1] \end{aligned}\tag{21}$$

where $\theta = (\rho,\lambda,\phi)^T$.

Denoted by $Z^{\theta_i}$ and $Z_1^{\theta_i}$ first derivatives of $Z$ and $Z_1$ with aspect to any parameter $\theta_i$ of the MINCOMPE distribution. The components of the unit score function $U = (U_\rho, U_\lambda, U_\phi)^T$ is given by

$$U_\rho = -n\frac{Z^\rho}{Z}+\sum_{i=0}^{n}\frac{Z_1^\rho}{Z_1}+\sum_{i=0}^{n}\frac{E[M_1]^\phi}{E[M_1]}, \tag{22}$$

and

$$U_\lambda = \sum_{i=0}^{n}\frac{Z_1^\lambda}{Z_1}+\sum_{i=0}^{n}\frac{E[M_1]^\lambda}{E[M_1]}, \tag{23}$$

and

$$U_\phi = -n\frac{Z^\phi}{Z}+\sum_{i=0}^{n}\frac{Z_1^\phi}{Z_1}+\sum_{i=0}^{n}\frac{E[M_1]^\rho}{E[M_1]}. \tag{24}$$

The numerical computation of the above moments can be easily performed in software packages such as R and Matlab. Numerical maximization of the log-likelihood function is performed with the RS method [6] in the gamlss package. These methods were discussed in detail in [7], and [4].

We show the log-likelihood functions for other models in corollaries below.

*Corollary 4:*

Where $\phi = 0$, the minimum service time had MINGE distribution and the log-likelihood function is accorded by

$$\begin{aligned}\ell(\rho,\lambda) &= n\lambda-\sum_{i=0}^{n}\lambda y+n\log\lambda(1-\rho) \\ &- 2\sum_{i=0}^{n}\log(1-\rho e^{-\lambda y_i}). \end{aligned}\tag{25}$$

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*Corollary 5:* Where $\phi = 1$, the minimum service time has MINPE distribution and the log-likelihood function is given by

$$\begin{aligned}\ell(\rho,\lambda) &= n\log(\rho\lambda) - n\rho - \sum_{i=0}^{n}\lambda y_i \\ &+ \rho\sum_{i=0}^{n} e^{\lambda y_i} - n\log(1-e^{\rho}). \qquad (26)\end{aligned}$$

*Corollary 6:* Where $\phi \to \infty$, the minimum service time has MINBE distribution and the log-likelihood function is given by

$$\ell(\rho,\lambda) = n\log\lambda\rho - \sum_{i=1}^{n}\lambda y_i - n\log(1+\rho). \qquad (27)$$

### C. Numerical Results

The numerical results are important to describe the behavior of the model and its applicability in different situations. We presented three pieces of data: the simulated data and two real data; data from a Brazilian supermarket checkout and data from the access to a website.

*1) Simulation:* For the simulated data, we have chosen $M/M/1$ model [8]. The aim was to look for the data where few customers remained in the queue. This particular set of data was then used to test the new distribution when the pressure parameter took the value $\phi = 0$. In this case, the system was not accelerated and the service rate was independent of the state of the system. Therefore, the $M/M/1$ model was simulated with the intensive traffic $\rho = 0.9$ with the arrival rate 0.9. We have stablished 1,000 arrivals as the ending point of the simulation. The proposal was to adjust the empirical models and it was based on the comparison between the observed and predicted values. The simplest way to make this comparison is graphically, which consists of comparing the reliability function to the Kaplan-Meier estimator. Thus, Figure 1 shows the behavior of the MINGE distribution it is compared with the Kaplan Meier estimates [9] for the simulated data with $\rho = 0.9$. Clearly, the MINGE distribution yields a close concordance with the Kaplan-Meier estimates.

*2) Real Data:*

- To begin with,the express checkout in the supermarket real data was analyzed. It is often felt that, the minimum service time is one of the key points in order to establish customer loyalty. Therefore, supermarkets use a variety of methods to reduce the service time at checkouts. The most traditional method for example, is the express lines. In the express lines the amount of items which customers can bring to an express checkout counter is limited. When analysing the supermarket checkouts, a particular express checkout presented a similar behavior of the system studied; the service was very fast and many customers entered directely into the service. The remaining number of customers in the queue was insignificant. We have used this data to test the MINGE distribution. The Kolmogorov-Smirnov test was used to prove that the interarrival times and the service times presented an exponential distributions. A total sample of 85 customers were observed with intensity traffic $\rho = 0.816$ where the mean service was 1.12 minutes. The MINGE distribution was used when the service was independent of the state of the system. The server was able to absorve all the works and it was not necessary to adjust the system. Figure 2 shows the MINGE distribution and the Kaplan-Meier estimates. Indeed, the MINGE distribution has a close concordance with the Kaplan-Meier estimates. The maximum likelihood estimates are given by $\hat{\rho} = 0.876$, $\hat{\lambda} = 0.90$ minutes and the mean $E(Y) = 1.908$ minutes.
- Finally, we analysed accesses to the website "Tendencias Profissionais" [10]. A survey with 26 questions was alocated on this website. Data collection

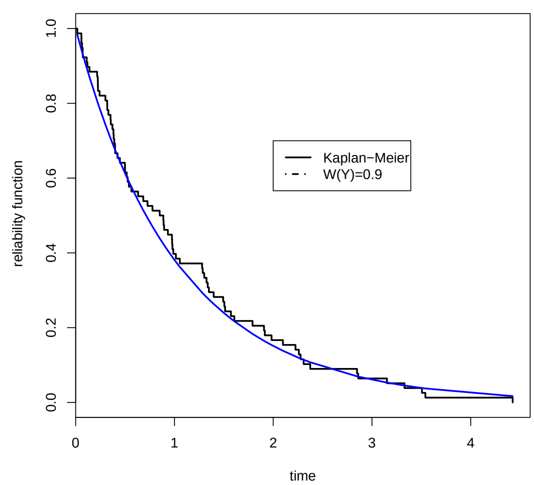


Figure 1: Kaplan-Meier estimates and reliability function $W(y)$ for $\rho = 0.9$.

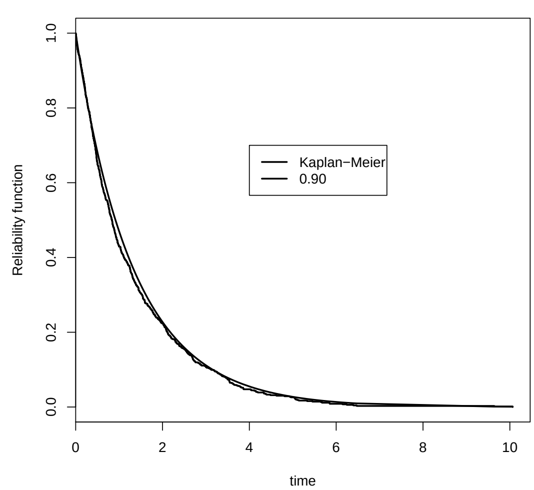


Figure 2: Kaplan-Meier estimates and reliability function $W_Y(y)$ for $\hat{\rho} = 0.876$.

started on 20 October, 2010 and it was available for 20 days. As a general rule, the internet allows the rapid dissemination of information. The survey was distributed through social networks and 1,000 emails were sent to the main communication agencies in Brazil. On the website ”Tendencias Profissionais”, the arrival time of customers was registered. A sequence of the arrival time was observed. The suitability of the exponential distribution was tested for interarrival times $(Y)$. The Kolmogorov-Smirnov test was used, therefore $D_{max} < D_n^{\alpha}$ was obtained and $D_{max} = 0.114$ and $D_n^{\alpha} = 0.122$ was the critical value. Moreover, the suitability of the Poisson model for the number of customers (M) was tested. A new test for the Poisson distribution [8]. The new test takes into account the non-homogeneity of the process as well as the under-dispertion of data. Therefore, the new test in which $T_{new} = 4\sum_{n=20}^{i=1}(\lambda_i - \bar{\lambda})^2 = 0.011$, where $\lambda_i$ is the arrival rate per day and $\bar{\lambda}$ is the average arrival rate of 20 days of observation. If $T_{new} > \chi^2_{n-1,1-\alpha}$, the hypothesis $H_0 M \sim Poisson(\lambda)$ is rejected. In this case, it was obtained that $T_{new} = 0.10 < \chi^2_{20-1,0.05}$, the hypothesis $H_0$ was rejected. Thus, the use of MINPE distribution was justified. Figure 3 shows the MINPE distribution and the Kaplan-Meier estimates. The MINPE distribution yields a close concordance with the Kaplan-Meier estimates. Moreover, the mean rate for answering the survey was 6.5 minutes and the minimum time was 1.2 minutes. In addition, the maximum likelihood estimates were given by $\hat{\rho} = 0.98$, $\hat{\lambda} = 0.449$ minutes and $E(Y) = 1.08$ minutes

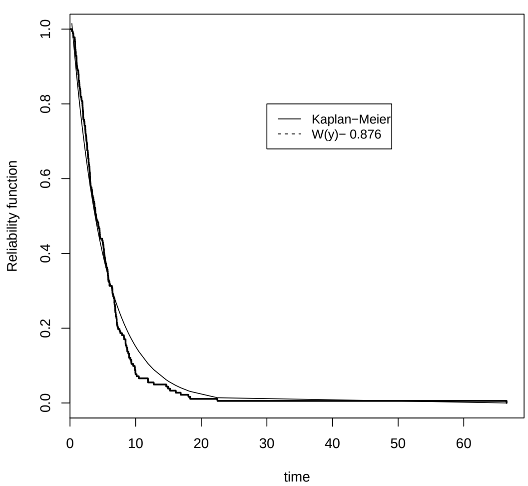


Figure 3: Kaplan-Meier estimates and reliability function $W(y)$ for $\hat{\rho} = 0.98$.

## II. Final Remarks

In this paper, we proposed a distribution for the minimum service model with service rate dependent on the state of the system called the Minimum-Conway-Maxwell-Poisson-exponential distribution and denoted by MINCOMPE distribution. This distribution describes the service, not considering the number of customers. In addition, there is a dependence between interarrival times and service times. In other words, the service is attached to the arrival and the interarrival time is the same as the service time. Hence, when the customer arrives in the system, he enters into the service directly. Therefore, it is necessary to have an adjustement mechanism in order to restablish the balance of the system. As a result, the service rate increases and/or new channels of the service can be opened. We studied three situations for the server. Firstly, the pressure parameter took on the zero value, $\phi = 0$ and in this case, the server dit not accelerate and the service was independent from the state of the system. Afterwards, the pressure parameter took on the value of one, $\phi = 1$, accelerating the server and opening new service channels. Finally, the server increased even more the service rate and the pressure parameter assumed the value infinity. The MINCOMPE distribution generalizes other usual distributions for each variation of the pressure parameter, such as the Minimum-geometric-exponential, Minimum-Poisson-exponential and Minimum-Bernoulli-exponential. The properties of the proposed distribution were discussed, including a formal proof of its pdf and moments. An estimation of the parameters was obtained by the maximum likelihood method. In order to illustrate the model. Real and simulated data were set as illustrations of how to fit the MINCOMPE distribution. To conclude, we believe that the MINCOMPE distribution has a practical approach within a service model with the state dependent service rate. In addition, this can be applied to various practical situations.

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