Semi-Analytic Modelling of Stratified Flows

Theory and Applications

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Abstract— **The problem on non-homogeneous shear flows over the rough terrain is considered semi-analytically. Mathematical modelling of these flows is interesting due to important applications in meteorology and oceanography. Known results presumably refer to the wave phenomena appearing in the flows over single bell-shaped obstacle. Presently, most intrinsic problem is to describe a complicated interference patterns which can be forced by multiple-ridged topography. In this paper, a non-linear model of a stratified flow over combined obstacle is constructed under the small amplitude assumption for the topography. Attention is focused on the stationary wave patterns formed directly above the hill range. Wave solutions corresponding to the topography with a finite number of peaks are calculated. These solutions predict rigorously the splitting of a near-field flow to the separate wave zones having different spatial scales.**

Keywords-semi-analytical mode; perturbation method; stratified flows.

I. INTRODUCTION

Modeling of stratified flows plays a significant role in several environmental disciplines, especially in meteorology and oceanography [1][2]. Internal waves in the non-homogeneous atmosphere and ocean are generated frequently from the interaction of the mean flow with orographic obstacles, such as mountains and submarine ridges. Lee waves arise downstream of the obstacle under appropriate upwind conditions. These waves possess horizontal lengths amounting to tens of kilometers and typical magnitudes of vertical displacement are on the order of hundreds of meters. Therefore, they can present a hazard to air traffic and sub-sea operations.

The theory of lee waves deals with the mathematical model of inviscid incompressible non-homogeneous fluid. Long's model is based on the linear Helmholtz equation for a steady stream function which should satisfy appropriate boundary conditions and radiation condition at infinity (we refer to the paper [3] for a mathematical details of the Long's theory).

This linear partial differential equation arises as a leading-order approximation to more general non-linear Dubreil-Jacotin – Long equation of stratified fluid [4]. Despite the linearity, explicit analytic solutions are known only for the simplest topographies, such as a single semi-circular obstacle [5][6]. Numerical solutions to Long's theory also encounter substantial difficulties due to the specific form of a boundary condition for arbitrary topography [7][8].

We develop a semi-analytical approach [9] involving the von Mises transformation of both dependent and independent variables in the non-linear version of the Dubreil-Jacotin – Long equation. The main idea of this method is to satisfy the exact topography condition by solving approximate equations in an auxiliary rectangular domain. The impact of the non-linearity is analyzed by the perturbation procedure with a small parameter which characterizes typical height of an obstacle. Our attention is focused on the fragmentation effects for the near-field wave patterns forced by the rough topography of finite extension.

This paper is organized as follows: Section II describes the mathematical setup, including the formulation of a basic model; Section III characterizes the perturbation procedure combined with the Fourier method to construct an analytical solution; Section IV illustrates preliminary results of numerical modelling of stratified flows. Discussion and conclusions are presented in Section V.

II. MATHEMATICAL FORMULATION

The mathematical model of a steady stratified 2-D flow over an uneven bottom is formulated as the boundary value problem for a second-order elliptic partial differential equation, i.e., DJL (Dubreil-Jacotin – Long) equation, which has the form

$$\psi_{xx} + \psi_{yy} + \lambda(\psi - y) = \frac{1}{2}\sigma\left(\psi_x^2 + \psi_y^2 - 1\right), (-\infty < x < \infty),$$

$$\psi(x, \alpha y_0(x)) = 0, \quad \psi(x,1) = 1, \qquad (1)$$

$$\psi(x, y) \to y \quad (x \to -\infty).$$

Here, the unknown function $\psi(x, y)$ is the stream function and the constants $\sigma > 0$ and $\lambda > 0$ are the Boussinesq parameter and the inverse densimetric Froude number, respectively. These dimensionless parameters are defined by the formulae

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$$\sigma = \frac{N^2 h}{g}, \quad \lambda = \frac{\sigma g h}{c^2},$$

where g is the gravity acceleration, h is the total depth of the stratified fluid layer, N is the constant Brunt–Väisälä frequency (i.e., the buoyancy frequency of air or ocean water), and c is the speed of the far-upstream flow. Quantity σ determines the slope of the density profile for a uniformly stratified fluid being at rest and λ indicates the value of the sub- or super-criticality of the upstream flow with respect to the phase speed of infinitesimal internal waves. Finally, the small parameter α characterizes a typical height of the bottom topography $y = \alpha y_0(x)$ towered above the ground level *y=0*.

From the mathematical point of view, problem (1) is a non-linear eigenvalue problem with spectral parameter λ as the bifurcation parameter. A non-trivial wave solution can bifurcate from the wave-less regimes if the magnitude of topography α is sufficiently small. Bifurcation occurs by λ belonging to a continuous spectrum of linear waves (see Fig. 1).

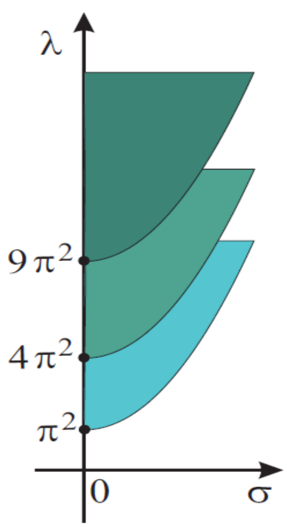

Figure 1. The spectrum of normal modes.

A parametric range of *m*-modal lee waves is formed by the sub-critical spectral domain determined by the inequalities

$$\pi^2 m^2 + \frac{1}{4}\sigma^2 < \lambda < \pi^2 (m+1)^2 + \frac{1}{4}\sigma^2.$$

Based on that, a non-trivial solution should have periodic asymptotics with respect to spatial variable x far-downstream the obstacle. In that sense, problem (1) can be also considered as the non-linear diffraction problem. The main difficulty is that the amplitude of stationary wave forced behind the obstacle is unknown ***a priory***.

III. Analytic Solution

We apply a semi-analytical approach involving von Mises transformation of the DJL equation. Namely, we seek the streamlines in the form $y = Y(x,\psi)$ with a new independent (x,ψ) -variables so that the flow domain transforms to the unit strip $0 < \psi < 1$. This transformation does not permit overhanging streamlines which are typical for developed structures of lee waves of large amplitude. However, such a geometric assumption allows to satisfy the exact topographic boundary condition at leading order approximate solution.

By given σ and λ, we construct solution $Y(x,\psi)$ with small α as the power series

$$Y(x,\psi) = \psi + \alpha w_0(x,\psi) + \alpha^2 w_1(x,\psi) + \ldots$$

The leading-order coefficient w_0 should satisfy both the homogeneous linear partial differential equation and the non-homogeneous topographic boundary condition at the bottom line $\psi = 0$; such that we have a linear elliptic boundary value problem

$$w_{0xx} + w_{0\psi\psi} - \sigma w_{0\psi} + \lambda w_0 = 0, \quad 0 < \psi < 1,$$
$$w_0(x,0) = y_0(x), \quad w_0(x,1) = 0.$$

Certainly, this approximation corresponds to the familiar equations of the Long's model. The difference is that here this model involves a more convenient topographic boundary condition than usual.

For Froude number λ belonging to the sub-critical spectral range of *m*-modal lee waves, we obtain the leading-order solution as follows:

$$w_0(x,\psi) = e^{\frac{\sigma\psi}{2}} \left\{ W(\psi) y_0(\psi) + \sum_{n=1}^{\infty} w_0^{(n)}(x) \sin \pi n \psi \right\}. \quad (2)$$

Here, function $W(\psi)$ corresponds to the hydrostatic mode of the flow over uneven bottom. This function is given by the formula

$$W(\psi) = \frac{\sin k_0 (1-\psi)}{\sin k_0}, \quad k_0 = \sqrt{\lambda - \frac{\sigma^2}{4}}.$$

The wave number k_0 here is real while parameter λ is sub-critical and Fourier-coefficients $w_0^{(n)}(x)$ are determined by the shape of the obstacle only.

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Similarly, we can construct the second-order solution, which takes into account a non-linear correction of the flow by solving the boundary value problem

$$w_{1xx} + w_{1\psi\psi} - \sigma w_{1\psi} + \lambda w_1 = f(w_0), \quad 0 < \psi < 1,$$
$$w_1(x,0) = w_1(x,1) = 0.$$

Here, the right-hand side involves the non-linearity, which has the form

$$f = \left(w_x w_\psi\right)_x + \frac{1}{2}\left(w_x^2 + 3w_\psi^2\right)_\psi - \frac{1}{2}\sigma\left(w_x^2 + 3w_\psi^2\right).$$

For known w_0, solution w_1 can also be presented as an infinite modal Fourier-series (2) with separated variables x and ψ.

IV. Examples of Calculated Wave Patterns

Wolfram Mathematica® [10] was used at all stages of semi-analytic calculations. Symbolic computer algebra was applied to present truncated solution (2), which provides fast convergence. Most of the numerical simulations used the series for coefficients w_0 and w_1 with ten basic harmonics. Computational flow domain involved the discretization with 50 points in horizontal direction x and 10 points in ψ. The calculation of a non-linear second-order solution used a numerical result for a leading-order linear solution presented by 6-order interpolation splines. Multiple series of calculations were carried out on the two node computer cluster at Novosibirsk State University.

Figure 2 and Figure 3 demonstrate some examples of the calculated 1-mode lee wave patterns, which appear in the flow region over the finite number of obstacles. Parameters α and λ are taken as α =0.04 and λ=15 in all cases. The Boussinesq parameter σ is chosen as σ =0.2.

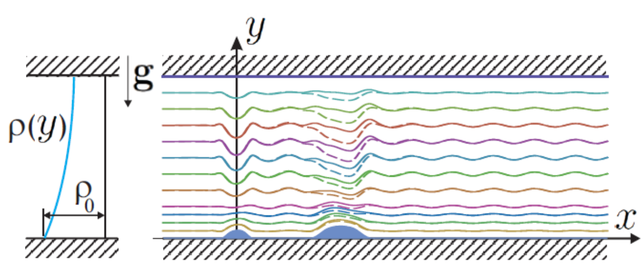

Figure 2. Lee waves over two bell-shaped obstacles.

Figure 2 illustrates the influence of the non-linearity on the stratified flow formed over the double bell-shaped topography. The non-linearity essentially corrects only the wave column immediately above the second obstacle but the downstream wave-tail still remains non-perturbed behind the obstacles.

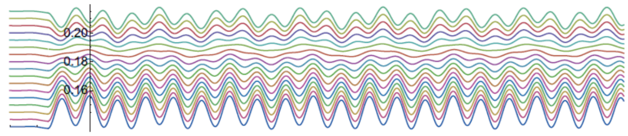

Figure 3. Separation of stratified flow.

Figure 3 illustrates an interesting effect of the wave interference, which can be observed in the near-bottom region of the boundary-trapped waves over a multi-hill topography. The near-bottom flow is clearly separated from the upper region of slowly modulated waves having the maximum amplitude at the mid-height of the fluid layer. The separation between lower and upper wave zones occurs at the height predicted by the zero points of the hydrostatic mode.

V. Conclusion and Future Work

In this paper, we outline semi-analytic approach assigned to simulate atmospheric stratified flows over combined 2D topography. The method exploits calculation the Fourier series which present modal decomposition of the waves forced by localized multi-bumped obstacle. Such an analytic solution of fluid mechanics is constructed by the perturbation procedure taking into account the non-linearity of mathematical problem. As noted, the limitations can arise by modelling the flows with overturning streamlines above the sharp-crested terrain with high peaks. However, the method seems to be well-conditioned while it predicts realistically the fragmentation of wave patterns due to interference of lee waves from adjacent ridges. Preliminary results demonstrate the ability to provide fast computations of the flows even for irregular-shaped topography. From a geophysical viewpoint, an efficient method is also needed to provide accurate computation of the fronts of separated wave zones. This work is in progress now. Next technical steps will operate with obtained Fourier solutions in order to evaluate analytically the impact of modal decomposition on the fragmentation effect observed by numerical experiments. Furthermore, we also plan to extend this method to modelling of stratified air flows over topography in local regions being of interest from meteorological viewpoint. By that, important issue is to estimate the computation requirements which are asserted to the algorithm by such a real topography.

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