Fuzzy Weight Representation for Double Inner Dependence Structure in 4 Levels AHP

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***Abstract* - The inner dependence Analytic Hierarchy Process (AHP) is useful for the cases in which criteria or/and alternatives are not independent enough and related to modeling and optimization. However, using the original AHP or inner dependence AHP may cause results that cannot have enough reliability because of the inconsistency of the comparison matrix as data. In such cases, fuzzy representation for weighting criteria or/and alternatives using results from sensitivity analysis is useful. In this research, we first define fuzzy local weights of criteria and alternatives. Moreover, via fuzzy sets, overall weights for double inner dependence structure AHP in 4 levels are obtained.**

Keywords* - *AHP; fuzzy sets; sensitivity analysis.

I. INTRODUCTION

The Analytic Hierarchy Process (AHP) proposed by T.L. Saaty in 1977 [1] is widely used in decision making, because it reflects humans feelings naturally. A normal AHP assumes independence among criteria and alternatives, although it is difficult to choose enough independent elements. The inner dependence method AHP [2] is used to solve this problem even for criteria or alternatives having dependence.

On the other hand, the comparison data matrix may not have enough consistency when AHP is applied because, for instance, a problem may contain too many criteria or alternatives for decision making. It means that answers from decision-makers, i.e., components of the matrix, do not have enough reliability. They may be too ambiguous or too fuzzy [3][5]. To avoid this problem, we usually have to revise again, but it takes a lot of time and costs.

Then, we consider that weights should also have ambiguity or fuzziness. Therefore, it is necessary to represent these weights using fuzzy sets. In our research, we first applied sensitivity analysis to normal AHP to analyze how much the components of a pairwise comparison matrix influence the weight or consistency of a matrix, and proposed new fuzzy weight representation for criteria and alternatives in normal AHP. Then, a representation of criteria weights for inner dependence AHP was proposed using L-R fuzzy numbers [4]. In the next step, we started to deal with double inner dependence structure [6] and their fuzzy weight.

We now consider fuzziness for double inner dependence [7][8] (among actors and criteria, respectively) when a comparison matrix among elements does not have enough consistency in 4 levels problem (object, actors, criteria and alternatives).

In Sections 2 and 3, we introduce the inner dependence AHP, consistency index, and sensitivity analyses for AHP. Then, in Section 4, we define fuzzy weights for double inner dependence structure, and Section 5 is a summary.

II. CONSISTENCY AND INNER DEPENDENCE

A. Process of Normal AHP

(Process 1) Representation of structure by a hierarchy. The problem under consideration can be represented in a hierarchical structure. At the middle levels, there are multiple criteria. Alternative elements are put at the lowest level of the hierarchy.

(Process 2) Paired comparison between elements at each level. A pairwise comparison matrix A is created from a decision maker's answers. Let n be the number of elements at a certain level, the upper triangular components of the matrix a_{ij} ($i< j = 1,\ldots,n$) are 9, 8, .. , 2, 1, 1/2, …, or 1/9. These denote intensities of importance from element i to j. The lower triangular components a_{ji} are described with reciprocal numbers, for diagonal elements, let $a_{ii} = 1$.

(Process 3) Calculations of weight at each level. The weights of the elements, which represent grades of importance among each element, are calculated from the pairwise comparison matrix. The eigenvector that corresponds to a positive eigenvalue of the matrix is used in calculations throughout in the paper.

(Process 4) Priority of an alternative by a composition of weights. With repetition of composition of weights, the overall weights of the alternative, which are the priorities of the alternatives with respect to the overall objective, are finally found**.**

B. Consistency

Since components of the comparison matrix are obtained by comparisons between two elements, coherent consistency is not guaranteed. In AHP, the consistency of the comparison matrix A is measured by the following consistency index (C.I.)

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$$\text{C.I.} = \frac{\lambda_A - n}{n-1}, \tag{1}$$

where n is the order of comparison matrix A, and λ_A is its maximum eigenvalue (Frobenius root).

If the value of C.I. becomes smaller, then the degree of consistency becomes higher, and vice versa. The comparison matrix is consistent if the following holds.

$$\text{C.I.} \leq 0.1 \tag{2}$$

C. Inner Dependence Structure

The normal AHP ordinarily assumes independence among criteria and alternatives, although it is difficult to choose enough independent elements. The dependency means some kind of interaction among the elements. Inner dependence AHP [2] is used to solve this type of problem even for criteria or alternatives having dependence.

In the method, using a dependency matrix $F=\{ f_{ij} \}$, we can calculate modified weights $\boldsymbol{w}^{(n)}$ as follows,

$$\boldsymbol{w}^{(n)}=F\boldsymbol{w} \tag{3}$$

where $\boldsymbol{w}$ represents weights from independent criteria or alternatives, i.e., normal weights of normal AHP and dependency matrix F is consist of eigenvectors of influence matrices showing dependency among criteria or alternatives.

If there is dependence in both lower levels, i.e., not only among criteria but also among alternatives, we call such kind of structure "double inner dependence". In the double inner dependence structure, we have to calculate modified weights of criteria and alternatives, $\boldsymbol{w}^{(n)}$ and $\boldsymbol{u_i}^{(n)}$. Then we composite these 2 modified weights to obtain overall weights of alternative k, $v_k^{(n)}$ as follow:

$$v_k^{(n)} = \sum_i^m w_i^{(n)} u_{ik}^{(n)} \tag{4}$$

where m is the number of criteria.

Also, using the same steps again, we can composite weights of "triple inner dependence" structure, in the case when there is dependency in the 3 lower levels, i.e., not only among alternatives and 1 level criteria but also 2 levels of criteria.

III. Sensitivity Analyses

When we use AHP in some applications, it often occurs that a comparison matrix is not consistent or that there is not great difference among the overall weights of the alternatives. In these cases, it is very important to investigate how components of the pairwise comparison matrix influence its consistency or the weights. In this study, we use a method that some of the present authors have proposed before. It evaluates a fluctuation of the consistency index and the weights when the comparison matrix is perturbed. It is useful because it does not change the structure of the data.

Since the pairwise comparison matrix is a positive square matrix, Perron-Frobenius theorem holds. From Perron-Frobenius theorem, the following theorem about a perturbed comparison matrix holds.

Theorem 1 *Let* $A = (a_{ij})$, $(i, j = 1,\dots, n)$ *denote a comparison matrix and let* $A(\varepsilon) = A+ \varepsilon D_A$, $D_A=(a_{ij}d_{ij})$ *denote a matrix that has been perturbed. Let* λ_A *be the Frobenius root of A,* $\boldsymbol{w}$ *be the eigenvector corresponding to* λ_A, *and* $\boldsymbol{v}$ *be the eigenvector corresponding to the Frobenius root of A'. Then, a Frobenius root* $\lambda(\varepsilon)$ *of* $A(\varepsilon)$ *and a corresponding eigenvector* $\boldsymbol{w}(\varepsilon)$ *can be expressed as follows*

$$\lambda(\varepsilon) = \lambda_A + \varepsilon\lambda^{(1)} + o(\varepsilon), \tag{5}$$

$$\boldsymbol{w}(\varepsilon) = \boldsymbol{w} + \varepsilon\boldsymbol{w}^{(1)} + \boldsymbol{o}(\varepsilon), \tag{6}$$

where

$$\lambda^{(1)} = \frac{\boldsymbol{v}' D_A \boldsymbol{w}}{\boldsymbol{v}' \boldsymbol{w}}, \tag{7}$$

$\boldsymbol{w}^{(1)}$ *is an n-dimension vector that satisfies*

$$(A - \lambda_A I)\boldsymbol{w}^{(1)} = -(D_A - \lambda^{(1)} I)\boldsymbol{w}\ , \tag{8}$$

where $\boldsymbol{o}(\varepsilon)$ *denotes an n-dimension vector in which all components are* $o(\varepsilon)$.

About a fluctuation of the consistency index, the following corollaries hold.

Corollary 1 *Using appropriate* g_{ij}, *we can represent the consistency index* C.I.(ε) *of the perturbed comparison matrix* $A(\varepsilon)$ *as follows*

$$\text{C.I.}(\varepsilon) = \text{C.I.} + \varepsilon \sum_i^n \sum_j^n g_{ij} d_{ij} + o(\varepsilon). \tag{9}$$

To see g_{ij} in (9) in Corollary 1, we can determine how the components of a comparison matrix impart influence on its consistency.

Corollary 2 *Using appropriate* $h_{ij}^{(k)}$, *we can represent the fluctuation* $\boldsymbol{w}^{(1)}=(w_k^{(1)})$ *of the weight (i.e., the eigenvector corresponding to the Frobenius root) as follows*

$$w_k^{(1)} = \sum_i^n \sum_j^n h_{ij}^{(k)} d_{ij}. \tag{10}$$

Then, we can evaluate how the components of a comparison matrix impart influence on the weights, to see $h_{ij}^{(k)}$ in (10).

Proofs of these corollaries are shown in [4].

IV. Fuzzy Weights Representations

When a comparison matrix has poor consistency (i.e., 0.1<C.I.<0.2), comparison matrix components are considered to be fuzzy because they are results from human fuzzy judgment. Weights should therefore be treated as fuzzy numbers [5][6].

Definition 1 (fuzzy weight) Let $w_k^{(n)}$ be a crisp weight of criterion or alternative k of inner dependence model, and g_{ij} | $h_{ij}^{(k)}$| denote the coefficients found in Corollary 1 and 2. If 0.1<C.I.<0.2, then a fuzzy weight $\tilde{w}_k$ is defined by

$$\tilde{w}_k = (w_k, \alpha_k, \beta_k)_{LR} \tag{11}$$

$$\alpha_k = \text{C.I.}\sum_i^n \sum_j^n s(-, h_{kij}) g_{ij} \mid h_{kij} \mid, \tag{12}$$

$$\beta_k = \text{C.I.}\sum_i^n \sum_j^n s(+, h_{kij}) g_{ij} \mid h_{kij} \mid, \tag{13}$$

Then, we assume about double inner dependence structure in 4 levels problem. For example, above of the criteria level there might be actor's level for decision. "Leisure in holiday with family" may have 4 family actors {father, mother, older child A, younger child B}, 4 criteria {popularity, good for rain, fatigue, expense} and 4 alternatives {theme park, indoor theme park, cinema, zoo}. They may have dependency structure at 2 in 4 levels and inconsistency in some levels.

Even in these cases, we can define overall weights of alternatives with fuzzy representation using sensitivity analysis. Let the modified local weight of a actors, $\boldsymbol{x}^{(n)} = (x_i^{(n)})$, $i = 1, \dots, l$, using dependency matrices F_P, modified fuzzy weighs of criteria with only respect to actor i, $\tilde{\boldsymbol{w}}_i^{(n)} = (\tilde{w}_{ij}^{(n)})$, $i = 1, \dots, l$, $j = 1, \dots, m$ using dependency matrix F_C, and weights of alternatives with only respect to criterion j, $\boldsymbol{u}_j = (u_{jk})$, $j = 1, \dots, m$, $k = 1, \dots, m$. We can define the modified fuzzy weight

$$\tilde{w}_{ij}^{(n)} = (w_{ij}^{(n)}, \alpha_{ij}^{(n)}, \beta_{ij}^{(n)})_{LR} \tag{14}$$

$$\boldsymbol{x}^{(n)} = (x_i^{(n)}) = F_P \boldsymbol{x} \tag{15}$$

$$\boldsymbol{w}_i^{(n)} = (w_{ij}^{(n)}) = F_C \boldsymbol{w}_i \tag{16}$$

$\boldsymbol{w}_i$ is crisp weights of criteria with only respect to actor i, and α_{ij}, β_{ij} are calculated from a result of sensitivity analysis (details are shown in [6]) .

At last, fuzzy overall weights of alternative k can be calculated as follows:

$$\tilde{v}_k^{(n)} = \sum_i^l \sum_j^m x_i^{(n)} \tilde{w}_{ij}^{(n)} u_{jk} \tag{17}$$

If there is also inconsistency in actor level, using fuzzy weight $\tilde{x}_i^{(n)}$ instead of crisp $x_i^{(n)}$, fuzzy overall weights of alternative k can be calculated as follows:

$$\tilde{v}_k^{(n)} = \sum_i^l \sum_j^m \tilde{x}_i^{(n)} \otimes \tilde{w}_{ij}^{(n)} u_{jk} \tag{18}$$

where $\otimes$ denotes, fuzzy multiplication defined by extension principal.

V. Conclusion and future work

There are many cases in which data of AHP does not have enough reliability. For these cases, we propose fuzzy weight representation and compositions for double inner dependence in 4 levels AHP using sensitivity analysis. Our approach can show how to represent weights and is efficient to investigate how the result of AHP has fuzziness even if data are not enough consistent or reliable.

In the next step, we must find better fuzzy multiplication for composition fuzzy weights.

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