Periodic Solution of a Discretized Age-Dependent Model with a Dominant Age Class

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Abstract—**A delay differential equation for the population size is derived from an age-dependent model with a dominant age class. This equation is provided with impulse conditions and its discrete-time counterpart is constructed using the semi-discretization method. Sufficient conditions for the existence of a periodic solution of the resulting difference problem are found by Mawhin's continuation theorem.**

Keywords-age-dependent model; impulse effect; discrete-time equation; periodic solution.

I. Introduction

Many evolutionary processes in nature are characterized by the fact that at certain instants of time they experience a rapid change of their states. This leads to the investigation of differential equations and systems with discontinuous trajectories, or with impulse effect, called for brevity impulsive differential equations and systems [1][2]. The theory of the impulsive differential equations is one of the attractive branches of differential equations which has extensive realistic mathematical modelling applications in physics, chemistry, engineering, and biological and medical sciences.

A classical problem of the qualitative theory of differential equations is the existence of periodic (or almost periodic) solutions. Numerous references on this matter concerning differential equations with delay and impulsive differential equations can be found in [3].

In [4], an age-dependent model with a dominant age class was considered. In a special case the total population size satisfies a delay differential equation. Sufficient conditions for the existence of a periodic solution of this equation satisfying appropriate impulse conditions were presented.

A brief survey is given in Section II of the present paper. In Section III, we obtain a discrete counterpart of the problem using the semi-discretization method. Finally, in Section IV, we find sufficient conditions for the existence of a periodic solution of the resulting discrete problem using Mawhin's continuation theorem [5, p. 40].

II. Preliminaries

The following model is described in the papers of T. Kostova [6], T. Kostova and F. Milner [7], where the existence of oscillatory solutions is proved.

For two fixed ages σ_1, σ_2 such that $0 \le \sigma_1 < \sigma_2 < \infty$, the age distribution $u(a,t)$ of a population is considered, where a is the age and t the time, with dynamics described by the following integro-differential equation with age-boundary condition in integral form,

$$\begin{cases} \dfrac{\partial u}{\partial a} + \dfrac{\partial u}{\partial t} = -\delta(a,Q)u(a,t), & a,t > 0, \\ u(0,t) = \displaystyle\int_0^\infty \beta(a,Q)u(a,t)\,da, & t \ge 0, \\ u(a,0) = u_0(a), & a \ge 0, \end{cases} \tag{1}$$

where

$$Q = Q(t) = \int_{\sigma_1}^{\sigma_2} u(a,t)\,da$$

is the *dominant age cohort* size and $\delta(a,Q)$ and $\beta(a,Q)$ are, respectively, the age-specific death rate and birth modulus when the dominant age group is of size Q. It is assumed that δ, β and u_0 are nonnegative, and that u_0 is integrable (so that the initial population is finite). This model is a generalization of the classical one of Gurtin and MacCamy [8], which is obtained by setting $\sigma_1 = 0$ and $\sigma_2 = \infty$.

Further on, in [6][7], the special case

$$\beta(a,Q) = \begin{cases} \beta(Q), & a \in [\sigma_1, \sigma_2], \\ 0 & \text{otherwise}, \end{cases}$$

is considered. This means that the dominant age class is the only one capable of having offspring, i.e., births are possible only in the age interval $[\sigma_1, \sigma_2]$ and the fertility rate depends just on the size of the dominant age group itself (and not on the age within the group). Moreover, $\beta(Q) \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and the mortality rate $\delta > 0$ is assumed constant.Then, for the total population size,

$$P(t) = \int_0^\infty u(a,t)\,da,$$

the equation

$$\dot{P} + \delta P = \beta(Q)Q \tag{2}$$

is derived, where

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$$Q(t) = P(t-\sigma_1)e^{-\sigma_1\delta} - (t-\sigma_2)e^{-\sigma_2\delta} \qquad (3)$$

for $t > \sigma_2$. Thus, for $t > \sigma_2$, $P(t)$ satisfies a nonlinear scalar delay equation (2) with Q given by (3), while for $t \in [0, \sigma_2]$ $Q(t)$ and eventually $P(t)$ can be expressed in terms of the initial function $u_0(a)$ of the age-dependent model (1). Thus we find the initial function $P_0(t), t \in [0, \sigma_2]$ of the above mentioned delay equation.

We fix a number $\omega > 0$ much larger than the age σ_2, and try to obtain an ω -periodic regime of the population size by means of impulsive perturbations for a suitably chosen initial function u_0. More precisely, suppose that at certain moments t_k such that $t_{k+p} = t_k + \omega$ for all $k \in \mathbb{Z}$, the population size $P(t)$ is abruptly changed, while (2) with (3) is assumed to hold for all $t \in \mathbb{R}, t \neq t_k$. We normalize the quantities in (2) as follows:

$$s = \frac{t}{\omega},\ \ \Pi(s) = P(\omega s),\ \ D = \omega\delta,\ \ B(Q) = \omega\beta(Q).$$

Henceforth, we write again t, $\delta$and β instead of s, D and B, respectively, x instead of Π, and $h = \sigma_2/\omega$ will be the small parameter, while the still smaller quantity σ_1/ω will be assumed 0, for the sake of simplicity. We suppose that the time interval between two successive abrupt changes (impulse effects) $t_{k+1} - t_k$ is large in comparison with the "age" h for all $k \in \mathbb{Z}$, and look for 1--periodic solutions of the problem

$$\dot{x}(t) = -\delta x(t) + R\big(x(t), x(t-h)\big),\ \ t \neq t_k, \qquad (4)$$
$$\Delta x(t_k) = -B_k x(t_k) + a_k,\ \ k \in \mathbb{Z}, \qquad (5)$$

where $\Delta x(t_k) \equiv x(t_k + 0) - x(t_k - 0)$ is the magnitude of the impulse effect at the moment t_k, $x(t_k) \equiv x(t_k - 0)$, a_k, B_k are positive constants satisfying $a_{k+p} = a_k$, $B_{k+p} = B_k$ $(k \in \mathbb{Z})$, $R\big(x(t), x(t-h)\big) = \beta\big(Q(t)\big)Q(t)$, $Q(t) = x(t) - x(t-h)e^{-h\delta}$, $0 = t_0 < t_1 < \cdots < t_{p-1} < t_p = 1$. We can consider (4) for $t > 0$, the impulse conditions (5) for $k \geq 0$, with initial condition

$$x(s) = \phi(s)\ \ \text{for}\ \ s \in [-1,0], \qquad (6)$$

where the initial function $\phi(s)$ is piecewise continuous with possible points of discontinuity of the first kind at t_{-p+1}, t_{-p+2}, ... , t_{-1}. To find a 1-periodic solution of problem (4), (5) means to determine the initial function $\phi(s)$ so that the solution of the initial value problem (4), (5), (6) is 1-periodic.

III. STATEMENT OF THE PROBLEM

We suppose that the period ω has been chosen so that $\omega = N\sigma_2$ for a positive integer N, thus $h = 1/N$. We assume N so large that

$$h < \min_{k=\overline{1,p}} (t_{k+1} - t_k).$$

Then, each interval $[nh, (n+1)h]$ contains at most one instant of impulse effect t_k.

For convenience, we denote $n = [t/h]$, the greatest integer in t/h, and $n_k = [t_k/h]$. Clearly, we will have $n_{k+p} = n_k + N$ for all $k \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$, $n \neq n_k$. This means that the interval $[nh, (n+1)h]$ contains no instant of impulse effect t_k. We approximate the differential equation (4) on the interval $[nh, (n+1)h]$ by

$$\dot{x}(t) + \delta x(t) = R\left(x(nh), x\big((n-1)h\big)\right).$$

We multiply both sides of this equation by $e^{\delta t}$ and integrate over the interval $[nh, (n+1)h]$. Thus we obtain

$$\begin{aligned} x\big((n+1)h\big) - x(nh) &= -\big(1 - e^{-\delta/N}\big)x(nh) \\ &+ \frac{1 - e^{-\delta/N}}{\delta} R\left(x(nh), x\big((n-1)h\big)\right). \end{aligned} \qquad (7)$$

Henceforth, by abuse of notation, we write $x(n) = x(nh)$ and redefine $\Delta x(n) = x(n+1) - x(n)$ $(n \in \mathbb{Z})$. Now, (7) takes the form

$$\begin{aligned} \Delta x(n) &= -\big(1 - e^{-\delta/N}\big)x(n) \\ &+ \tfrac{1 - e^{-\delta/N}}{\delta} R\big(x(n), x(n-1)\big). \end{aligned} \qquad (8)$$

Next, for $n = n_k$, the interval $[nh, (n+1)h]$ contains the instant of impulse effect t_k. On this interval, we approximate the impulse conditions (5) by

$$\Delta x(n_k) = -B_k x(n_k) + a_k,\ \ k \in \mathbb{Z}. \qquad (9)$$

The difference system (8), (9) can be written in operator form as

$$\Delta x = Hx, \qquad (10)$$

where

$$\begin{aligned} (Hx)(n) &= -\big(1 - e^{-\delta/N}\big)x(n) \\ &+ \frac{1 - e^{-\delta/N}}{\delta} R\big(x(n), x(n-1)\big),\ \ n \neq n_k, \\ (Hx)(n_k) &= -B_k x(n_k) + a_k,\ \ k \in \mathbb{Z}. \end{aligned} \qquad (11)$$

We can consider the system (10) for $n \geq 0$, with initial conditions

$$x(\ell) = \phi(\ell)\ \ \text{for}\ \ \ell = 0, -1, \dots, -N, \qquad (12)$$

where $\phi(\ell)$, $\ell = 0, -1, \dots, -N$, is a given initial vector. To find an N-periodic solution of system (10) means to determine the initial vector $\phi(\ell)$ so that the solution of the initial value problem (10), (12) is N-periodic.

IV. MAIN RESULT

First, we introduce some notations:

$$A := \sum_{k=0}^{p-1} a_k, \quad B := \sum_{k=0}^{p-1} Ba_k, \quad D := (N-p)\left(1 - e^{-\frac{\delta}{N}}\right),$$
$$I_N := \{0,1,\dots,N-1\}, \qquad \mathfrak{I}_N := I_N \backslash \{n_k\}_{k=0}^{p-1}.$$

Next, we formulate some assumptions:

A1. There exists a constant $K > 0$ such that $|\beta(Q)| \le K$ for any $Q \in \mathbb{R}$.

A2. $\quad D + B - \dfrac{2DK}{\delta} - (D+B)\left(D + B + \dfrac{2DK}{\delta}\right) > 0.$

Remark 1. Assumption **A2** may seem quite complicated. We show that it is easy to satisfy. If we denote $y = D + B$, $z = \frac{2DK}{\delta}$, then assumption **A2** takes the form

$$y - z - y^2 - yz > 0, \text{ i.e.,} \ z < \frac{y(1-y)}{1+y}.$$

The right-hand side of the last inequality is positive for $0 < y < 1$, it achieves its maximum value $3 - 2\sqrt{2} \approx 0.18$ for $y = \sqrt{2} - 1 \approx 0.41$. Thus, it suffices to choose $D + B = 0.41$ and $\frac{2DK}{\delta} < 0.18$.

Remark 2. The inequality

$$D + B - \frac{DK}{\delta}\left(1 - e^{-\frac{\delta}{N}}\right) > 0 \tag{13}$$

follows from assumption **A2**. In fact,

$$\begin{aligned} D + B - \frac{DK}{\delta}\left(1 - e^{-\frac{\delta}{N}}\right) &> D + B - \frac{DK}{\delta} \\ = \left[D + B - \frac{2DK}{\delta} - (D+B)\left(D + B + \frac{2DK}{\delta}\right)\right] & \\ + \left[\frac{DK}{\delta} + (D+B)\left(D + B + \frac{2DK}{\delta}\right)\right] &> 0. \end{aligned}$$

Now, we can state our main result as the following theorem.

Theorem 1. *Suppose that assumptions* **A1**, **A2** *hold. Then,* (10) *has at least one* N*-periodic solution*.

Proof. We shall prove Theorem 1 using Mawhin's continuation theorem [5, p. 40]. To state this theorem, we need some preliminaries (see [9][10]).

Let $\mathbb{X}, \mathbb{Y}$ be real Banach spaces, $L: \operatorname{Dom} L \subset \mathbb{X} \to \mathbb{Y}$ be a linear mapping, and $H: \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in $\mathbb{Y}$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P_1: \mathbb{X} \to \mathbb{X}$ and $P_2: \mathbb{Y} \to \mathbb{Y}$ such that $\operatorname{Im} P_1 = \operatorname{Ker} L$, $\operatorname{Ker} P_2 = \operatorname{Im} L = \operatorname{Im}(I - P_2)$, then the mapping $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P_1}: (I - P_1)\mathbb{X} \to \operatorname{Im} L$ is invertible. We denote the inverse of this mapping by K_{P_1}. If Ω is an open bounded subset of $\mathbb{X}$, the mapping H will be called L-compact on $\overline{\Omega}$ if $P_2 H(\overline{\Omega})$ is bounded and $K_{P_1}(I - P_2)H: \overline{\Omega} \to \mathbb{X}$ is compact. Since $\operatorname{Im} P_2$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J: \operatorname{Im} P_2 \to \operatorname{Ker} L$.

Now, Mawhin's continuation theorem can be stated as follows.

Lemma 1. *Let L be a Fredholm mapping of index zero, let $\Omega \subset \mathbb{X}$ be an open bounded set, and let $H: \mathbb{X} \to \mathbb{Y}$ be a continuous operator, which is L-compact on $\overline{\Omega}$. Assume that the following conditions hold*:
(*a*) *for each* $\lambda \in (0,1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \ne \lambda Hx$;
(*b*) *for each* $x \in \partial\Omega \cap \operatorname{Ker} L$, $P_2 Hx \ne 0$;
(*c*) $\deg(JP_2H, \Omega \cap \operatorname{Ker} L, 0) \ne 0$, *where* $\deg(\cdot)$ *is the Brouwer degree*.

Then, the equation $Lx = Hx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

Before we proceed further, we shall recall the definition of Brouwer degree [11].

Suppose that M and N are two oriented differentiable manifolds of dimension n (without boundary), with M compact and N connected, and suppose that $f: M \to N$ is a differentiable mapping. Let $Df(x)$ denote the differential mapping at the point $x \in M$, that is, the linear mapping $Df(x): T_x M \to T_{f(x)} N$. Let $\operatorname{sign} Df(x)$ denote the sign of the determinant of $Df(x)$. That is, the sign is positive if f preserves orientation, and negative if f reverses orientation.

Definition 1. Let $y \in N$ be a regular value, then we define the *Brouwer degree* (or just *degree*) of f by

$$\deg f = \sum_{x \in f^{-1}(y)} \operatorname{sign} Df(x).$$

It can be shown that the degree does not depend on the regular value y that we pick, so that $\deg f$ is well defined.

Note that this degree coincides with the degree as defined for maps of spheres.

Let us choose

$$\mathbb{X} = \mathbb{Y} = \{x(n): x(n+N) = x(n), n \in \mathbb{Z}\}.$$

If we define $\|x\| = \max_{n \in I_N} |x(n)|$, then $\mathbb{X}$ is a Banach space with the norm $\|\cdot\|$. For $x \in \mathbb{X}$, let Hx be defined by (11), $Lx = \Delta x$ and $P_1 x = P_2 x = \frac{1}{N}\sum_{n=0}^{N-1} x(n)$. Then, $\operatorname{Ker} L = \{x \in \mathbb{X}: x = c \in \mathbb{R}\}$ (independent of n), $\operatorname{Im} L = \{x \in \mathbb{X}: \sum_{n=0}^{N-1} x(n) = 0\}$ is a closed set in $\mathbb{X}$, and $\operatorname{codim} L = 1$. Thus, L is a Fredholm mapping of index zero. It is easy to see that P_1 and P_1 are continuous projectors and $\operatorname{Im} P_1 = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} P_2 = \operatorname{Im}(I - P_2)$, and H is L-compact on $\overline{\Omega}$ for any bounded set $\Omega \subset \mathbb{X}$. Moreover, in condition (*c*) of Lemma 1 the isomorphism J can be taken as the identity operator I.

Now, we will derive some estimates for the solutions x of the operator equation $Lx = \lambda Hx$ for $\lambda \in (0,1)$, that is,

$$\Delta x(n) = \lambda (Hx)(n), \ \ n \in I_N.$$

First, from (11) for $n \neq n_k$, we obtain

$$\begin{aligned} |\Delta x(n)| &\leq \left(1 - e^{-\frac{\delta}{N}}\right)|x(n)| + \frac{1 - e^{-\frac{\delta}{N}}}{\delta}|\beta(Q)|\,|Q| \\ &\leq \left(1 - e^{-\frac{\delta}{N}}\right)\left\{\|x\| + \frac{K}{\delta}\left|x(n) - x(n-1)e^{-\delta/N}\right|\right\} \\ &\leq \left(1 - e^{-\frac{\delta}{N}}\right)\left(1 + \frac{2K}{\delta}\right)\|x\|. \end{aligned}$$

Similarly, for $n = n_k$, we have

$$|\Delta x(n_k)| \leq B_k \|x\| + a_k.$$

From the above inequalities, we obtain

$$\sum_{n=0}^{N-1} |\Delta x(n)| \leq (N - p)\left(1 - e^{-\frac{\delta}{N}}\right)\left(1 + \frac{2K}{\delta}\right)\|x\| + \sum_{k=0}^{p-1} B_k \|x\| + \sum_{k=0}^{p-1} a_k,$$

or

$$\sum_{n=0}^{N-1} |\Delta x(n)| \leq \left[D\left(1 + \frac{2K}{\delta}\right) + B\right]\|x\| + A. \tag{14}$$

Adding together all equations of (8), (9) for $n \in I_N$, we obtain

$$\left(1 - e^{-\frac{\delta}{N}}\right)\sum_{n \in \Im_N} x(n) + \sum_{k=0}^{p-1} B_k x(n_k) = \frac{1 - e^{-\frac{\delta}{N}}}{\delta}\sum_{n \in \Im_N} R\big(x(n), x(n-1)\big) + \sum_{k=0}^{p-1} a_k.$$

Then, as above, we obtain

$$\left|\left(1 - e^{-\frac{\delta}{N}}\right)\sum_{n \in \Im_N} x(n) + \sum_{k=0}^{p-1} B_k x(n_k)\right| \leq D\frac{2K}{\delta}\|x\| + A. \tag{15}$$

Now, we shall use the following lemma (see [12] [13]).

Lemma 2. *Let $v: \mathbb{Z} \to \mathbb{R}$ be N-periodic, i.e., $v(n + N) = v(N)$ for any $n \in \mathbb{Z}$. Then, for any fixed $v_1, v_2 \in I_N$ and any $n \in \mathbb{Z}$, we have*

$$v(v_2) - \sum_{k=0}^{N-1} |v(k+1) - v(k)| \leq v(n) \leq v(v_1) + \sum_{k=0}^{N-1} |v(k+1) - v(k)|.$$

According to Lemma 2, for arbitrary $n, v_1, v_2 \in I_N$, we have

$$x(v_2) - \sum_{n=0}^{N-1} |\Delta x(n)| \leq x(n) \leq x(v_1) + \sum_{n=0}^{N-1} |\Delta x(n)|.$$

We multiply these inequalities by $1 - e^{-\delta/N}$ for $n \neq n_k$ or B_k for $n = n_k$, and sum up over I_N to obtain

$$\begin{aligned} (D + B)x(v_2) - (D + B)\sum_{n=0}^{N-1} |\Delta x(n)| &\leq \left(1 - e^{-\frac{\delta}{N}}\right)\sum_{n \in \Im_N} x(n) + \sum_{k=0}^{p-1} B_k x(n_k) \\ &\leq (D + B)x(v_1) + (D + B)\sum_{n=0}^{N-1} |\Delta x(n)|. \end{aligned}$$

From the last two inequalities, we deduce

$$-x(v_1) \leq -\frac{\left(1 - e^{-\frac{\delta}{N}}\right)\sum_{n \in \Im_N} x(n) + \sum_{k=0}^{p-1} B_k x(n_k)}{D + B} + \sum_{n=0}^{N-1} |\Delta x(n)|,$$

$$x(v_2) \leq \frac{\left(1 - e^{-\frac{\delta}{N}}\right)\sum_{n \in \Im_N} x(n) + \sum_{k=0}^{p-1} B_k x(n_k)}{D + B} + \sum_{n=0}^{N-1} |\Delta x(n)|.$$

Let $|x(v_0)| = \|x\| = \max_{n \in I_N} |x(n)|$. If $x(v_0) \geq 0$, we choose $v_2 = v_0$. Then,

$$\begin{aligned} (D + B)\|x\| &= (D + B)x(v_2) \\ &\leq \left|\left(1 - e^{-\frac{\delta}{N}}\right)\sum_{n \in \Im_N} x(n) + \sum_{k=0}^{p-1} B_k x(n_k)\right| \\ &\quad + (D + B)\sum_{n=0}^{N-1} |\Delta x(n)|. \end{aligned}$$

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If $x(\nu_0) < 0$, we choose $\nu_2 = \nu_0$,

$$\begin{aligned}(D+B)\|x\| &= -(D+B)x(\nu_1)\\ &\le -\left(\left(1-e^{-\frac{\delta}{N}}\right)\sum_{n\in\Im_N} x(n) + \sum_{k=0}^{p-1} B_k x(n_k)\right)\\ &\quad +(D+B)\sum_{n=0}^{N-1}|\Delta x(n)|\\ &\le \left|\left(1-e^{-\frac{\delta}{N}}\right)\sum_{n\in\Im_N} x(n) + \sum_{k=0}^{p-1} B_k x(n_k)\right|\\ &\quad +(D+B)\sum_{n=0}^{N-1}|\Delta x(n)|.\end{aligned}$$

Thus, in both cases, we have

$$\begin{aligned}(D+B)\|x\| \le &\left|\left(1-e^{-\frac{\delta}{N}}\right)\sum_{n\in\Im_N} x(n) + \sum_{k=0}^{p-1} B_k x(n_k)\right|\\ &+(D+B)\sum_{n=0}^{N-1}|\Delta x(n)|.\end{aligned}$$

Making use of the estimates (14) and (15), we obtain

$$\begin{aligned}(D+B)\|x\| &\le D\frac{2K}{\delta}\|x\| + A\\ &\quad +(D+B)\left\{\left[D\left(1+\frac{2K}{\delta}\right)+B\right]\|x\| + A\right\}\\ &= \left\{D\frac{2K}{\delta} + (D+B)\left[D\left(1+\frac{2K}{\delta}\right)+B\right]\right\}\|x\|\\ &\quad +A(1+D+B),\end{aligned}$$

or

$$\begin{aligned}\left\{D+B-D\frac{2K}{\delta}-(D+B)\left(D+B+D\frac{2K}{\delta}\right)\right\}\|x\|\\ \le A(1+D+B).\end{aligned}$$

By virtue of assumption **A2**, the number

$$C^* := \frac{A(1+D+B)}{D+B-D\frac{2K}{\delta}-(D+B)\left(D+B+D\frac{2K}{\delta}\right)} > 0,$$

and each solution x of the operator equation $Lx = \lambda Hx$ for $\lambda \in (0,1)$ satisfies the inequality $\|x\| \le C^*$.

Now, we take $\Omega = \{x \in \mathbb{X}: \|x\| < C\}$, where $C > C^*$ will be chosen later. For $x \in \partial\Omega \cap \operatorname{Dom} L$, we have $\|x\| = C$, thus x cannot be a solution of $Lx = \lambda Hx$ for $\lambda \in (0,1)$. Obviously, Ω satisfies condition (*a*) of Lemma 1.

Now, let $x \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}$, i.e., x is a constant in $\mathbb{R}$ with $|x| = C$. For such x,

$$NP_2Hx = D\left[-x + \beta(x)x\left(1-e^{-\delta/N}\right)\right] - Bx + A,$$

and

$$N\|P_2Hx\| \ge \left[D + B - \frac{DK}{\delta}\left(1-e^{-\frac{\delta}{N}}\right)\right]C - A.$$

By inequality (13), we can choose $C > C^*$ so large that

$$\left[D + B - \frac{DK}{\delta}\left(1-e^{-\frac{\delta}{N}}\right)\right]C > A.$$

Hence, for $x \in \partial\Omega \cap \operatorname{Ker} L$, we have $N\|P_2Hx\| > 0$ and $P_2Hx \ne 0$, that is, condition (*b*) of Lemma 1 is satisfied.

To prove (*c*), we define the mapping

$$(P_2H)_\mu : \operatorname{Dom} L \times [0,1] \to \mathbb{X}$$

by

$$(P_2H)_\mu = \mu P_2\widetilde{H} + (1-\mu)P_2H,$$

where the operator $\widetilde{H}$ is defined by

$$\begin{aligned}\left(\widetilde{H}x\right)(n) &= -\left(1-e^{-\frac{\delta}{N}}\right)x(n),\ \ n \ne n_k,\\ \left(\widetilde{H}x\right)(n_k) &= -B_k x(n_k),\ \ k \in \mathbb{Z}.\end{aligned}$$

For $x \in \partial\Omega \cap \operatorname{Ker} L$, we have

$$\begin{aligned}N(P_2H)_\mu x = D\left[-x + (1-\mu)\beta(x)x\left(1-e^{-\delta/N}\right)\right]\\ -Bx + (1-\mu)A.\end{aligned}$$

As above, we obtain

$$N\left\|(P_2H)_\mu x\right\| \ge \left[D + B - \frac{DK}{\delta}\left(1-e^{-\frac{\delta}{N}}\right)\right]C - A > 0.$$

This means that $(P_2H)_\mu x$ for $x \in \partial\Omega \cap \operatorname{Ker} L$ and $\mu \in [0,1]$. From the homotopy invariance of the Brouwer degree, it follows that

$$\begin{aligned}&\deg(P_2H, \Omega \cap \operatorname{Ker} L, 0)\\ &= \deg\left(P_2\widetilde{H}, \Omega \cap \operatorname{Ker} L, 0\right) = -1 \ne 0.\end{aligned}$$

According to Lemma 1, (10) has at least one N-periodic solution. This completes the proof of Theorem 1. ■

V. CONCLUSIONS

In the present paper, we derived a delay differential equation for the population size from an age-dependent model with a dominant age class. We provided this equation with impulse conditions and constructed its discrete-time counterpart using the semi-discretization method. We found sufficient conditions for the existence of a periodic solution of the resulting difference problem, in the form of assumptions **A1** and **A2**, by Mawhin's continuation theorem. **A1** assumes boundedness of the birth modulus, while **A2** is a not too complicated algebraic equation. Similar methods can be used to find conditions for the existence of periodic solutions of equations arising in physics and chemistry.

REFERENCES

[1] V. Lakshmikantham, D. Bainov, and P. Simeonov, Theory of Impulsive Differential Equations, Series in Modern Applied Mathematics, vol. 6, Singapore: World Scientific, 1989, ISBN: 978-9971-5-0970-5 (hardcover), 978-981-4507-26-4 (ebook), 978-981-281-275-9 (institutional ebook).

[2] A. M. Samoilenko and N. A. Perestyuk, Impulsive Differential Equations, World Scientific Series on Nonlinear Science Series A, vol. 14, Singapore: World Scientific, 1995, ISBN: 978-281-02-2416-5 (harcover), 978-981-4499-82-8 (ebook), 978-981-279-866-4 (institutional ebook).

[3] D. D. Bainov and V. C. Covachev, "Periodic solutions of impulsive systems with a small delay," J. Phys. A: Math. Gen., vol. 27, pp. 5551–5563, 1994, ISSN: 0305-4470 (print), 1361-6447 (online).

[4] V. Covachev, "Periodicity of the population size of an age-dependent model with a dominant age class by means of impulsive perturbations," Int. J. Appl. Math. Comput. Sci., vol. 10, pp. 147–155, 2000, ISSN: 2083-8492.

[5] R. E. Gaines and J. L. Mawhin, Coincidence Degree and Nonlinear Differential Equations, Lecture Notes in Mathematics, vol. 568, Berlin: Springer, 1977, ISBN: 978-3-540-08067-1 (softcover), 978-3-540-37501-2 (ebook).

[6] T. Kostova, Oscillations in an Age-Dependent Model with a Dominant Age Class, Sofia: Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Preprint No. 5 (March 1995).

[7] T. Kostova and F. Milner, "An age-structured model of population dynamics with dominant ages, delayed behavior, and oscillations," Math. Popul. Stud., vol. 5, pp. 359–375, 1995, ISSN: 0889-8480.

[8] M. E. Gurtin, R. C. MacCamy, "Nonlinear age-dependent population dynamics," Arch. Ration. Mech. Anal., vol. 54, pp. 281–300, 1974, ISSN: 0003-9527 (print), 1432-0673 (online).

[9] H. Akça, V. Covachev, Z. Covacheva, and S. Mohamad, "Global exponential periodicity for the discrete analogue of an impulsive Hopfield neural network with finite distributed delays," Funct. Differ. Equ., vol.16, pp. 53–72, 2009, ISSN: 0793-1786.

[10] Y. Li, L. Zhao, and X. Chen, "Existence of periodic solutions for neural type cellular neural networks with delays," Appl. Math. Model., vol. 36, pp. 1173–1183, 2012, ISSN: 0307-904X.

[11] J. W. Milnor, Topology from the Differentiable Viewpoint, Princeton Landmarks in Mathematics, Princeton University Press, 1997, ISBN-13: 978-0691048338, ISBN-10: 0691048339.

[12] M. Fan and K. Wang, "Periodic solutions of a discrete time nonautonomous ratio-dependent predator-prey system," Math. Comput. Model., vol. 35, pp. 951–961, 2002, ISSN: 0895-7177.

[13] R. Xu, L. Chen, and F. Hao, "Periodic solutions of discrete-time Lotka-Volterra type food-chain model with delays," Appl. Math. Comput., vol. 171, pp. 91–103, 2005, ISSN: 0096-3003 (print), 1873-5649 (online).