On a Fuzzy AHP Weight for Partial Inner Dependence Structure

Shin-ichi Ohnishi, Takahiro Yamanoi
Faculty of Engineering
Hokkai-Gakuen University
Sapporo, Japan
email: {ohnishi, yamanoi}@hgu.jp

***Abstract* - In a field of decision making for systems that contains human beings, the Analytic Hierarchy Process (AHP) is widely employed. It elicits weights of criteria and alternatives that are independent enough of each other. For cases in which criteria are not independent enough, an extended inner dependence AHP is useful. In this paper, we investigate "partial inner dependence" structure, i.e., only some elements (proper subset) of the criteria are independent. For the partial inner dependence AHP, we propose a new kind of fuzzy weight representation that is valid even if a data matrix is not consistent or reliable enough. Although lots of kinds of fuzzy weight for AHP have been proposed, our new representation can be defined by using the results of two kinds of the sensitivity analyses and they are useful for partial inner dependence structure. We finally show a numerical example of the fuzzy weight for partial inner dependence AHP.**

Keywords - AHP; fuzzy set; sensitivity analysis.

I. INTRODUCTION

The Analytic Hierarchy Process (AHP) proposed by T.L. Saaty in 1977 [1] is widely used in decision making because it reflects humans feelings naturally. The normal AHP assumes independence among all criteria, although it is difficult to choose enough independent elements. The inner dependence AHP [2] is used to solve this kind of problem when criteria have dependency. However, inner dependence method requires dependency matrix for all elements even if some criteria are independent. In this research, we employ "partial inner dependence" structure. Our method divides a set of criteria to two subsets, such as a dependent part and an independent part, and then we can easily understand a relation among elements.

On the other hand, the comparison data matrix may not have enough consistency when AHP is applied, because, for instance, a problem may contain too many criteria to make decision. It means that answers from decision-makers, i.e., components of the matrix, do not have enough reliability. They may be too ambiguous or too fuzzy [3][5]. To avoid this issue, we usually have to revise again, but it takes a lot of time and costs. Then, we consider that weights should also have ambiguity or fuzziness. Therefore, it is necessary to represent these weights using fuzzy set.

In our research, we first apply sensitivity analysis to normal AHP to analyze how much the components of a pairwise comparison matrix influence the weight and/or consistency indices of the matrix. Next, we define new fuzzy weight representation of criteria for partial inner dependence AHP using L-R fuzzy numbers [4][6][7][8]. At last, we then propose overall fuzzy weight of alternatives when a comparison matrix among elements does not have enough consistency. Though a main idea of representing fuzzy weight for inner dependence structure was proposed by authors several years ago, we extend this representation to partial inner dependence.

In Sections 2 and 3, we introduce the partial inner dependence AHP, consistency index and sensitivity analyses for AHP. Then, in Section 4, we define fuzzy weight for partial inner dependence structure, Section 5 is a numerical example, and Section 6 is a summary.

II. CONSISTENCY AND INNER DEPENDENCE

In this section, we introduce the processes of the normal AHP, its consistency and inner dependence extension.

A. Normal AHP

Usually, the AHP consists of following 4 processes.

(Process 1) Representation of structure by a hierarchy. The problem under consideration can be represented in a hierarchical structure. At the middle levels, there are multiple criteria. Alternative elements are put at the lowest level of the hierarchy.

(Process 2) Paired comparison between elements at each level. A pairwise comparison matrix A is created from a decision maker's answers. Let n be the number of elements at a certain level, the upper triangular components of the matrix a_{ij} ($i < j = 1,\dots,n$) are 9, 8, .. , 2, 1, 1/2, …, or 1/9. These denote intensities of importance from element i to j. The lower triangular components a_{ji} are described with reciprocal numbers, for diagonal elements, let $a_{ii} = 1$.

(Process 3) Calculations of weight at each level. The weights of the elements, which represent grades of importance among each element, are calculated from the pairwise comparison matrix. The eigenvector that corresponds to a positive normalized (so as sum of components is 1) eigenvalue of the matrix is used in calculations throughout in the paper.

(Process 4) Priority of an alternative by a composition of weights. With repetition of composition of weights, the overall weights of the alternative, which are the priorities of

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the alternatives with respect to the overall objective, are finally found.

B. Consistency

Since components of the comparison matrix are obtained by comparisons between two elements, coherent consistency is not guaranteed. In AHP, the consistency of the comparison matrix A is measured by the following consistency index (C.I.)

$$\text{C.I.} = \frac{\lambda_A - n}{n-1}, \tag{1}$$

where n is the order of comparison matrix A, and λ_A is its maximum eigenvalue (Frobenius root).

If the value of C.I. becomes smaller, then the degree of consistency becomes higher, and vice versa. It is said that the comparison matrix is consistent if $\text{C.I.} \leq 0.1$.

C. Partial Inner Dependence Method

The normal AHP ordinarily assumes independency among criteria, although it is difficult to choose enough independent elements in practice. The dependency means some kind of interaction among the elements. Inner dependence AHP [2] is used to solve this type of problem even for the case that criteria have dependency.

In the inner dependence method, using a dependency matrix $F=\{ f_{ij} \}$, we can calculate modified weights $\boldsymbol{w}^{(n)}$ as follows,

$$\boldsymbol{w}^{(n)}=F\boldsymbol{w} \tag{2}$$

where $\boldsymbol{w}$ represents weights from independent criteria, i.e., normalized weight of normal AHP and dependency matrix F consists of eigenvectors of influence matrices that represent dependency among criteria. However, inner dependence method requires dependency matrix for all elements even if some criteria are independent. In this research, we employ "partial inner dependence" structure, and then we can easily understand a relation among elements.

In a partial inner dependence AHP, we can divide a criteria set $C = \{X_1, X_2, ..., X_n\}$ to two subsets, dependent part $C_a = \{X_1^{(a)}, X_2^{(a)}, ..., X_{n1}^{(a)}\}$ and independent part $C_b = \{X_1^{(b)}, X_2^{(b)}, ..., X_{n2}^{(b)}\}$, $n_1 + n_2 = n$, they are determined whether the element is independent criterion or not. Let weights of C_a be $\boldsymbol{w}^{(a)} = (w_{i_1}^{(a)}),\ i_1 = 1, \ldots, n_1$, and weight of C_b be $\boldsymbol{w}^{(b)} = (w_{i_2}^{(b)}),\ i_2 = 1, \ldots, n_2$.

First, we calculate modified weight of dependent criteria subset $\boldsymbol{w}^{(an)} = (w_{i_1}^{(an)})$, using dependency matrix F as follows:

$$\boldsymbol{w}^{(an)} = F\boldsymbol{w}^{(a)}. \tag{3}$$

Then, the partial crisp (i.e. not fuzzy yet) weight $\boldsymbol{w}^{(pn)} = (w_i^{(pn)})$, $i = 1, ..., n$ is made by the following connection.

$$\boldsymbol{w}^{(pn)} = (w_1^{(an)}, \ldots, w_{n_1}^{(an)}, w_1^{(b)}, \ldots, w_{n_2}^{(b)}) \tag{4}$$

Using this modified criterion weight, we can easily calculate the priority of alternatives, i.e., overall weight of alternatives with respect to overall objective.

III. Sensitivity Analyses

When we use AHP in some applications, it often occurs that a comparison matrix is not consistent or that there is not great difference among the overall weights of the alternatives. In these cases, it is very important to investigate how components of the pairwise comparison matrix influence its consistency or the weights. In this study, we use a method that some of the present authors have proposed before. It evaluates a fluctuation of the consistency index and the weights when the comparison matrix is perturbed. It is useful because it does not change the structure of the data.

Since the pairwise comparison matrix is a positive square matrix, Perron-Frobenius theorem holds. From Perron-Frobenius theorem, the following theorem about a perturbed comparison matrix holds.

Theorem 1 *Let* $A = (a_{ij})$, $(i, j = 1, \ldots, n)$ *denote a comparison matrix and let* $A(\varepsilon) = A + \varepsilon D_A$, $D_A = (a_{ij} d_{ij})$ *denote a matrix that has been perturbed. Let* λ_A *be the Frobenius root of* A, $\boldsymbol{w}$ *be the eigenvector corresponding to* λ_A, *and* $\boldsymbol{v}$ *be the eigenvector corresponding to the Frobenius root of transposed* A'. *Then, a Frobenius root* $\lambda(\varepsilon)$ *of* $A(\varepsilon)$ *and a corresponding eigenvector* $\boldsymbol{w}(\varepsilon)$ *can be expressed as follows*

$$\lambda(\varepsilon) = \lambda_A + \varepsilon\lambda^{(1)} + o(\varepsilon), \tag{5}$$

$$\boldsymbol{w}(\varepsilon) = \boldsymbol{w} + \varepsilon\boldsymbol{w}^{(1)} + \boldsymbol{o}(\varepsilon), \tag{6}$$

where

$$\lambda^{(1)} = \frac{\boldsymbol{v}' D_A \boldsymbol{w}}{\boldsymbol{v}' \boldsymbol{w}}, \tag{7}$$

$\boldsymbol{w}^{(1)}$ *is an n-dimension vector that satisfies*

$$(A - \lambda_A I)\boldsymbol{w}^{(1)} = -(D_A - \lambda^{(1)} I)\boldsymbol{w}, \tag{8}$$

where $\boldsymbol{o}(\varepsilon)$ *denotes an n-dimension vector in which all components are* $o(\varepsilon)$.

About a fluctuation of the consistency index, the following corollaries hold.

Corollary 1 *Using appropriate* g_{ij}*, we can represent the consistency index* C.I.(ε) *of the perturbed comparison matrix* $A(\varepsilon)$ *as follows*

$$\text{C.I.}(\varepsilon) = \text{C.I.} + \varepsilon \sum_i^n \sum_j^n g_{ij} d_{ij} + o(\varepsilon). \tag{9}$$

To see g_{ij} in (9) in Corollary 1, we can determine how the components of a comparison matrix impart influence on its consistency.

Corollary 2 *Using appropriate* $h_{ij}^{(k)}$*, we can represent the fluctuation* $\boldsymbol{w}^{(1)}=(w_k^{(1)})$ *of the weight* (*i.e., the eigenvector corresponding to the Frobenius root*) *as follows*

$$w_k^{(1)} = \sum_i^n \sum_j^n h_{ij}^{(k)} d_{ij}. \tag{10}$$

Then, we can evaluate how the components of a comparison matrix impart influence on the weights, to see $h_{ij}^{(k)}$ in (10).

Proofs of these corollaries are shown in [4].

IV. Fuzzy Weight Representation

When a comparison matrix has poor consistency (i.e., 0.1<C.I.<0.2), components of the comparison matrix are considered to be fuzzy because they are results from human fuzzy judgment. Therefore, weight should be treated as fuzzy numbers [4][6].

Definition 1 (fuzzy weight) Let $w_k^{(pn)}$, $k = 1,...,n$, be a crisp weight of criterion k of partial inner dependence model, and g_{ij} | $h_{ij}^{(k)}$| denote the coefficients found in Corollary 1 and 2. If 0.1<C.I.<0.2, then a fuzzy weight of partial inner dependence criteria $\tilde{\boldsymbol{w}}^{(pn)} = (\tilde{w}_k^{(pn)}), k = 1,...,n$ can be defined by

$$\tilde{w}_k^{(pn)} = (w_k^{(pn)}, \alpha_k, \beta_k)_{LR} \tag{11}$$

where

$$\alpha_k = \text{C.I.} \sum_i^n \sum_j^n s(-, h_{ij}^{(k)}) g_{ij} \mid h_{ij}^{(k)} \mid, \tag{12}$$

$$\beta_k = \text{C.I.} \sum_i^n \sum_j^n s(+, h_{ij}^{(k)}) g_{ij} \mid h_{ij}^{(k)} \mid, \tag{13}$$

The definition above is an extension of fuzzy weight representation for independent structure [4]. Therefore, we can use this definition for dependent and independent elements part both.

Using the above definition, the overall fuzzy weight of alternative l ($l = 1,...,m$) can be calculated as follows:

$$\tilde{v}_l = \sum_k^n \tilde{w}_k^{(pn)} u_{kl} \tag{14}$$

where u_{kl}, $k = 1,...,n,\ l = 1,...,m$ is weight of the l-th alternatives with only respect to the criterion k.

V. Numerical Example

In this section, we show an example of “Leisure in holiday with family” having 4 criteria and 4 alternatives. Criteria are {popularity, good for rain (rain), fatigue, expense} and alternatives are {theme park (park), indoor theme park (indoor), cinema, zoo}.

Table I shows a comparison matrix of criteria and normal weight, where its consistency is not so good (C.I. >0.1). Then using results of sensitivity analyses of consistency and weights, we can calculate fuzzy weights. There is a partial inner dependence between only 2 criteria (popularity and expense). Dependency matrix between popularity and expense is shown in Table II. Therefore, using a dependency matrix of criteria (TABLE II) and results of sensitivity analyses, modified fuzzy weight are obtained as shown in TABLE III. For partial inner dependence structure, centers of popularity and expense are changed.

TABLE IV shows weights of alternatives with only respect to criteria and consistency index. There is consistency in all criteria. Finally, using composition (14), we evaluate overall fuzzy weights of alternatives in TABLE V.

TABLE I. Comparison Matrix

	popularity	rain	fatigue	excpense	*weight*
popularity	1.000	0.500	5.000	0.333	0.214
rain	2.000	1.000	2.000	0.333	0.226
fatigue	0.200	0.500	1.000	0.333	0.092
expense	3.000	3.000	3.000	1.000	0.469

C.I.= 0.134

TABLE II. Dependency Matrix of Criteria *F*

	popularity	excpense
popularity	0.800	0.400
expense	0.200	0.600

TABLE III. Modified Fuzzy Weight of Criteria.

	Center	*Spread(L)*	*Spread(R)*
popularity	0.358	*0.0079*	*0.0080*
rain	0.226	*0.0072*	*0.0071*
fatigue	0.092	*0.0052*	*0.0024*
expense	0.324	*0.0014*	*0.0043*

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TABLE IV. LOCAL WEIGHT OF ALTERNATIVES AND ITS CONSISTENCY

	popularity	rain	fatigue	excpense
park	0.333	0.206	0.112	0.134
indoor	0.105	0.165	0.305	0.313
cinema	0.154	0.499	0.517	0.435
zoo	0.408	0.130	0.067	0.113
C.I.	0.032	0.69	0.012	0.039

TABLE V. OVERALL FUZZY WEIGHT OF ALTERNATIVES

	Center	*Spread(L)*	*Spread(R)*
park	0.219	*0.0007*	*0.0007*
indoor	0.204	*0.0005*	*0.0005*
cinema	0.357	*0.0011*	*0.0011*
zoo	0.218	*0.0006*	*0.0006*

From overall weight of alternatives, we can make fuzzy decision in partial inner dependence structure.

VI. CONCLUSION AND FUTURE WORK

There are many cases in which data of AHP does not have enough consistency or reliability and structure of a problem does not contain complete independent criteria. For these cases, we propose a fuzzy weight representation and compositions for incomplete inner dependence structure using results of sensitivity analyses and fuzzy set. Although lots of kinds of fuzzy weight for AHP have been proposed, our new representation can be defined by using the results of two kinds of the sensitivity analyses and they are useful for partial inner dependence structure. Our example can not only show how to represent weight of criteria and alternatives, but also makes it possible to investigate how the result of AHP has fuzziness even if data are not consistent or reliable enough.

In the next step, we will compare the partial inner dependence AHP and the normal AHP with real data.

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