Wind and Wave Power Quality Estimation using a Lévy Statistical Analysis of the Wind Velocity

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Abstract—**The power quality of a wind turbine is determined by many factors but time-dependent variation in the wind velocity are arguably the most important. After a brief review of the statistics of typical wind speed data, a non-Gaussian model for the wind velocity is introduced that is based on a Lévy distribution. It is shown how this distribution can be used to derive a stochastic fractional diffusion equation for the wind velocity as a function of time whose solution is characterized by the Lévy index. A Lévy index numerical analysis is then performed on wind velocity data for both rural and urban areas where, in the latter case, the index has a larger value. Finally, empirical relationships are derived for the power output from a wind turbine in terms of the Lévy index using Betz law and for an idealized wave energy converter.**

Keywords-**Complex RIAS, Electrical power systems, Wind turbines, Stochastic wind velocity model, Non-Gaussian statistics, Lévy index, Quality control**

I. Introduction

Developing appropriate models for assessing and predicting the *quality of power* for any renewable energy source is important throughout the energy industry. Quality of power modeling is particularly important with regard to wind energy as the construction of new wind farms is growing rapidly compared with other renewable energy systems [1]. By 2030, it is estimated that up to 40% world energy supply will be based on renewable energy sources and in countries with an appropriate disposition to generating energy from wind, wave and tidal power such as the UK and Ireland, the percentage is expected to be much higher.

Quality of power modeling is often based on a statistical analysis of the available wind velocity data which is used to assess optimum regions for the construction of wind farms [2]. Although the power generated by a wind turbine is based on a range of design factors, the wind velocity as a primary factor since, from Betz law, the power P in Watts is given by [3]

$$P = \frac{1}{2}\alpha\rho A v^3 \qquad (1)$$

where v is the wind speed in metres per second (ms^{-1}), A is the area of the turbine in m^2, ρ is the density of air in kgm^{-3} and $\alpha < 0.593$ is the coefficient of performance. Although other physical factors such as air temperature and pressure,

angle of attack, etc. are important, the scaling law of the output power with regard to wind velocity (i.e., $P \propto v^3$) is the most significant feature for a given design of a wind turbine with a fixed area and coefficient of performance [4]. Thus, an understanding of the time variations in the wind velocity for a given geographical location is of paramount importance with regard to locating a wind farm and monitoring its performance in terms of the power quality. This requires stochastic models to be developed for the power output [5].

The acquisition of wind velocity data over different time intervals and localities is a common practice together with a routine statistical analysis of the data. The analysis is almost exclusively based on the assumption that time variations in the wind velocity are random Brownian processes and that the rate of change of velocity as a function of time is Gaussian distributed, i.e., the wind velocity conforms to a process of diffusion. However, this is not usually the case as discussed in the following section and in this paper we develop a non-Gaussian stochastic model for the wind velocity that is based on a Lévy distribution and a fractional diffusion equation. This allows us to analyze wind velocity in terms of the Lévy index and thereby yields an approach to assessing the quality of power for a wind turbine in terms of this index. We provide examples of wind velocity data that substantiate this approach and construct an empirical relationship for the power output from a wind turbine based on the Lévy index.

The structure of the paper is as follows: In Section II, we provide a brief overview on the statistics of typical wind velocity data emphasizing the non-Gaussian nature of the velocity gradient. In Section III, we provide an introduction to Lévy processes and introduce a specific Lévy distribution for characterizing the wind velocity (gradient). This section also considers the connection between using a Lévy distribution to characterize the wind velocity function and a description for this function in terms of a fractional diffusion equation whose solution provides an estimate for the wind velocity time series in terms of the Lévy index. Based on this result, in Section IV, we consider an analysis of the wind velocity data in terms of the Lévy index for a

moving window process and show how the Lévy index time signature and some of the statistical parameters associated with this signature can be used to characterize the wind velocity. Based on these results, in Section V, we derive an empirical relationship between the Lévy index for the wind velocity and an estimate for the power generated by a wind turbine using Betz law. An analogous relationship is derived in Section VI for an idealized wave energy converter. The relationships obtained between the Lévy index and the average (logarithmic) power output given in Sections V and VI represent an original contribution although no quantitative evaluation of these relationships is explored. In terms of a contribution to Resource Intensive Applications and Services, the results considered in this paper provide a frame work for estimating, monitoring and possibly predicting the power generated by wind and wave farms based on an analysis that is consistent with the known statistical characteristics of the wind velocity. The approach considered has applications is assessing the optimal location for the construction of wind and wave farms, for example, based on a non-standard, non-Gaussian statistical analysis when intensive data gathering and monitoring of environmental conditions is required.

II. STATISTICAL ANALYSIS OF THE WIND SPEED

Figure 1 shows a typical example plots of the wind velocity and wind direction as a function of time together with the associated histograms illustrating a marked difference in their statistical characteristics. This data shows wind velocities (in metres per second) and wind directions (in degrees) and consists of 8000 samples recorded at Dublin Airport, Ireland over intervals of 1 hour from 00:00:00 on 1 January 2008 to 06:00:00 on 29 November 2008. The The wind velocity $v(t)$ has a typical Rayleigh-type distribution with a mode of $5\mathrm{ms}^{-1}$ and a maximum wind velocity of $21.1\mathrm{ms}^{-1}$. The wind direction has a marked statistical bias toward higher angles with a primary mode of 240 degrees which is characteristic of the prevailing wind direction for the region.

Figure 2 compares the velocity gradient $d_t v(t)$ (which represents the force generated by the wind for a unit mass computed using a forward differencing scheme) with the output from a zero-mean Gaussian distributed random number stream. By comparing these signals, it is clear that the statistical characteristics of $d_t v(t)$ are not Gaussian. The plot of $d_v t$ obtained from the wind velocity data clearly shows that there are a number of rare but extreme events corresponding to short periods of time over which the change in wind velocity is relatively high. This leads to a distribution with a narrow width but longer tail when compared to a normal (Gaussian) distribution. Non-Gaussian distributions of this type are typical of Lévy processes which are discussed in the following section.
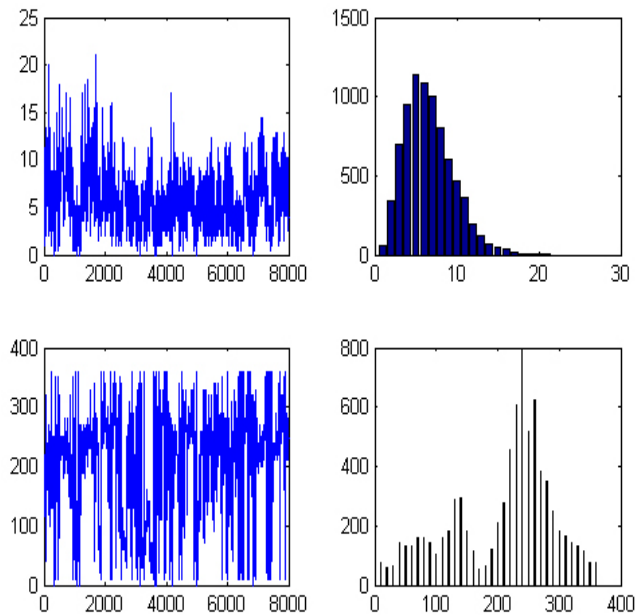

Figure 1. Plots of the wind velocity (top-left in metres per second) and wind direction (bottom-left in degrees) and the associated 22-bin and 360-bin histograms (top-right and bottom-right), respectively.
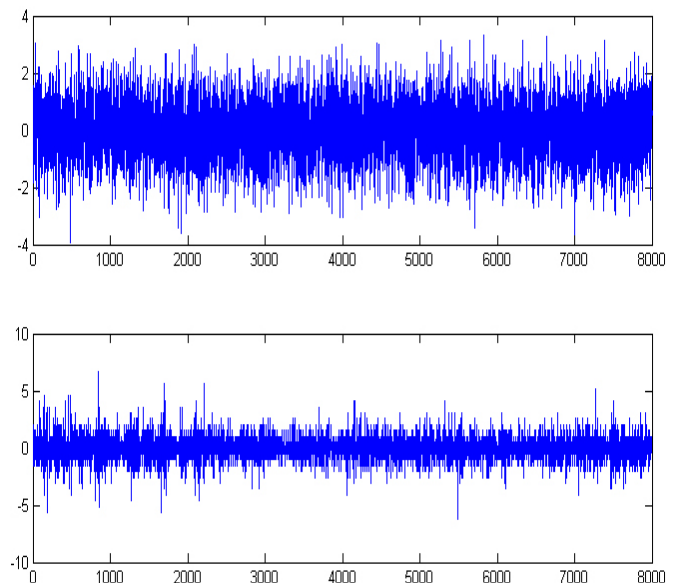

Figure 2. Plots of a zero-mean Gaussian distributed stochastic signal obtained using MATLAB V7 *randn* function (above) and the gradient of the wind velocity given in Figure 1 (below).

III. Lévy Processes

Lévy processes are random walks whose distribution has infinite moments. The statistics of (conventional) physical systems are usually concerned with stochastic fields that have PDFs (Probability Density Functions) where (at least) the first two moments (the mean and variance) are well defined and finite. Lévy statistics is concerned with stochastic processes where all the moments (starting with the mean) are infinite. Many distributions exist where the mean and variance are finite but are not representative of the process, e.g., the tail of the distribution is significant, where rare but extreme events occur. These distributions include Lévy distributions [6]. Lévy's original approach to deriving such distributions is based on the following question: Under what circumstances does the distribution associated with a random walk of a few steps look the same as the distribution after many steps (except for scaling)? This question is effectively the same as asking under what circumstances do we obtain a random walk that is statistically self-affine. The characteristic function $P(k)$ of such a distribution $p(x)$ was first shown by Lévy to be given by (for symmetric distributions only) [6]

$$P(k) = \exp(-a \mid k \mid^{\gamma}), \quad 0 < \gamma \leq 2 \qquad (2)$$

where a is a constant and γ is the Lévy index. For $\gamma \geq 2$, the second moment of the Lévy distribution exists and the sums of large numbers of independent trials are Gaussian distributed. If a stochastic process is characterized by a random walk with a step length distribution governed by $p(x)$ with $\gamma = 2$, then the result is normal (Gaussian) diffusion, i.e., a Brownian random walk process. For $\gamma < 2$ the second moment of this PDF (the mean square), diverges and the characteristic scale of the walk is lost. For values of γ between 0 and 2, Lévy's characteristic function corresponds to a PDF of the form

$$p(x) \sim \frac{1}{x^{1+\gamma}}, \quad x \to \infty \qquad (3)$$

Furthermore, Lévy processes characterized by a PDF of this type conform to a fractional diffusion equation as we shall now show [7].

The evolution equation for a random walk process that generates a macroscopic field denoted by $f(x,t)$ is given by

$$f(x, t + \tau) = f(x, t) \otimes_x p(x)$$

where $\otimes_x$ denotes the convolution integral over x and $p(x)$ is an arbitrary PDF. From the convolution theorem, in Fourier space, this equation becomes

$$F(k, t + \tau) = F(k, t)P(k)$$

where F and P are the Fourier transforms of f and p respectively. From equation (2), we note that

$$P(k) = 1 - a \mid k \mid^{\gamma}, \quad a \to 0$$

so that we can write

$$\frac{F(k, t + \tau) - F(k, t)}{\tau} \simeq -\frac{a}{\tau} \mid k \mid^{\gamma} F(k, t)$$

which for $\tau \to 0$ gives the fractional diffusion equation

$$\sigma \frac{\partial}{\partial t} f(x, t) = \frac{\partial^{\gamma}}{\partial x^{\gamma}} f(x, t), \quad \gamma \in (0, 2]$$

where $\sigma = \tau/a$ and we have used the result

$$\frac{\partial^{\gamma}}{\partial x^{\gamma}} f(x, t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \mid k \mid^{\gamma} F(k, t) \exp(ikx) dk$$

In terms of the application considered in this paper, the function $f(t)$ is taken to represent the wind force (the velocity gradient) with a non-Gaussian distributed time signature of the type illustrated in Figure 2 and taken to conform to the distribution in by equation (3) with a characteristic function given by equation (2) for $a \to 0$. However, since, for unit mass, $f(x, t) = \partial v(x, t)/\partial t$, we can consider the equation

$$\sigma \frac{\partial}{\partial t} v(x, t) = \frac{\partial^{\gamma}}{\partial x^{\gamma}} v(x, t), \quad \gamma \in (0, 2] \qquad (4)$$

for the wind velocity v. The solution to this equation with the singular initial condition $v(x, 0) = \delta(x)$ is given by

$$v(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx - t \mid k \mid^{\gamma} /\sigma) dk$$

which is itself Lévy distributed. This derivation of the fractional diffusion equation reveals its physical origin in terms of Lévy statistics.

For normalized units $\sigma = 1$ we consider equation (4) for a 'white noise' source function $n(t)$ and a spatial impulse function $-\delta(x)$ so that

$$\frac{\partial^{\gamma}}{\partial x^{\gamma}} v(x, t) - \frac{\partial}{\partial t} v(x, t) = -\delta(x)n(t), \quad \gamma \in (0, 2]$$

which, ignoring (complex) scaling constants, has the Green's function solution [8]

$$v(t) = \frac{1}{t^{1-1/\gamma}} \otimes_t n(t) \qquad (5)$$

where $\otimes_t$ denotes the convolution integral over t and $v(t) \equiv v(0, t)$. The function $v(t)$ has a Power Spectral Density Function (PSDF) given by (for scaling constant c)

$$\mid V(\omega) \mid^2 = \frac{c}{\mid \omega \mid^{2/\gamma}}$$

where

$$V(\omega) = \int_{-\infty}^{\infty} v(t) \exp(-i\omega t) dt$$

and a self-affine scaling relationship

$$\Pr[v(at)] = a^{1/\gamma} \Pr[v(t)]$$

for scaling parameter $a > 0$ where $\Pr[v(t)]$ denotes the PDF of $v(t)$. This scaling relationship means that the statistical characteristics of $v(t)$ are invariant of time accept for scaling factor $a^{1/\gamma}$. Thus, if $v(t)$ is taken to be the wind velocity as a function of time, then the statistical distribution of this function will be the same over different time scales whether, in practice, it is sampled in hours or seconds, for example.

IV. LÉVY INDEX ANALYSIS

The PSDF $\mid V(\omega) \mid^2$ provides a method of computing γ using the least squares method based on minimizing the error function

$$e(c, \gamma) = \|2 \ln \mid V(\omega) \mid - \ln c - 2\gamma^{-1} \ln \mid \omega \mid \|_2^2, \quad \omega > 0$$

Figures 3 and 4 show the computation of $\gamma(t)$ for a moving window of size 1024 elements. The accompanying tables (Table I and Table II) provide some basic statistical information with regard to $\gamma(t)$ for these data sets. Application of the Bera-Jarque parametric hypothesis test of composite normality is rejected (i.e., 'Composite Normality' is of type 'Reject') and thus $\gamma(t)$ is not normally distributed.
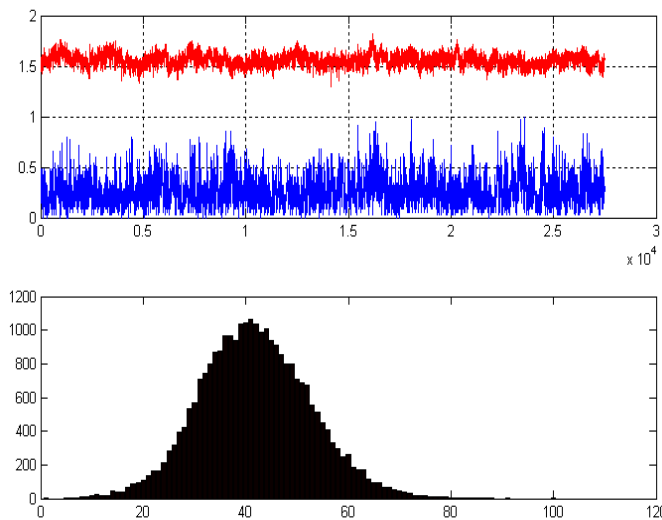

Figure 3. Cork Airport (12/11/2003-1/1/2007) for hourly (averaged) sampled data. Above: Normalized wind velocity data $v(t)$ (blue) and the Lévy index $\gamma(t)$ (red) for a look-back moving window of 1024 elements. Below: 100-bin histogram of $\gamma(t)$.

These result illustrates that the wind velocity function is a self-affine stochastic function with a mean Lévy index of ~ 1.5. Based on these results, Figure 5 shows a simulation of the wind velocity based on the computation of $v(t)$ in equation (5) for $\gamma = 1.5$. The simulation is based on transforming equation (5) into Fourier space and using a Discrete Fourier Transform. The function $n(t)$ is computed using MATLAB (V7) uniform random number generator *rand* for $seed = 1$.

Table I
STATISTICAL PARAMETERS ASSOCIATED WITH THE LÉVY INDEX FUNCTION GIVEN IN FIGURE 3.

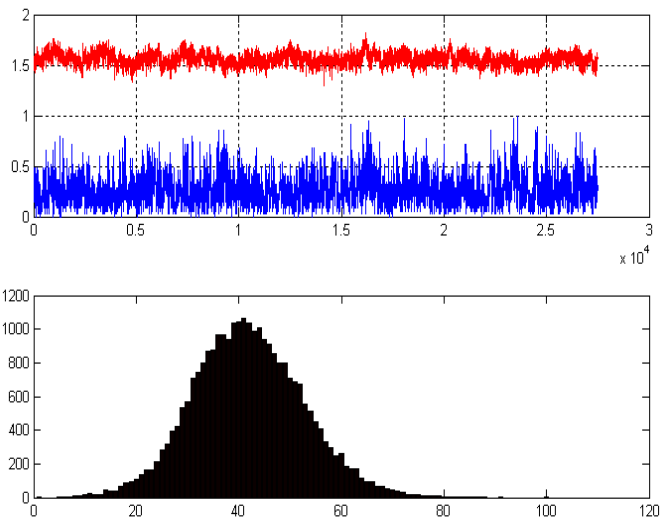| Statistical Parameter	Value for $\gamma(t)$
Minimum Value	1.3001
Maximum value	1.8142
Range	0.5141
Mean	1.5615
Median	1.5613
Standard Deviation	0.0569
Variance	0.0032
Skewness	0.0759
Kertosis	3.1966
Composite Normality	'Reject'

Figure 4. Knock Airport (12/11/2003-1/1/2005) for hourly (averaged) sampled data. Above: Normalised wind velocity data $v(t)$ (blue) and the Lévy index $\gamma(t)$ (red) for a look-back moving window of 1024 elements. Below: 100-bin histogram of $\gamma(t)$.

Table II
STATISTICAL PARAMETERS ASSOCIATED WITH THE LÉVY INDEX FUNCTION GIVEN IN FIGURE 4.

Statistical Parameter	Value for $\gamma(t)$
Minimum Value	1.3846
Maximum value	1.7600
Range	0.3754
Mean	1.5777
Median	1.5788
Standard Deviation	0.0510
Variance	0.0026
Skewness	-0.1538
Kertosis	3.0764
Composite Normality	'Reject'

The results given in Figure 3 and Figure 4 are for wind velocity data obtained in rural areas, i.e., at Cork and Knock airports, respectively. It is interesting to note that, in urban areas, the Lévy index may be expected to increase as a result of the further 'diffusion' of the wind velocity through 'random scattering' of the wind from buildings in the local vicinity when, according the model being considered, $\gamma \rightarrow 2$. An example of this is given in Figure 6 and Table III in which the average Lévy index is ~ 1.72 thereby confirming this expectation.

V. POWER QUALITY ESTIMATION FOR WIND ENERGY GENERATION

Given equation (1) and equation (5), we can obtain an expression for the power output by a wind turbine in terms of the Lévy index γ as a function of time. Let the noise function in equation (5) be a simple impulse at an instant in time so that $n(t) = \delta(t)$. Then

$$v(t) = \frac{1}{t^{1-1/\gamma}}$$

and, from equation (1),

$$P(t) = \frac{\beta}{t^{3(1-1/\gamma)}}$$

where $\beta = \alpha \rho A/2$ so that

$$\ln P(t) = \ln \beta - 3 \ln t + \frac{3}{\gamma} \ln t$$

Given that β is a constant, it is then clear that, for any time t, the magnitude of $\ln P$ is determined by γ^{-1}. In this sense, γ^{-1} is a coefficient of power quality as a function of time and we see that, according to this model, power output increases as γ decreases. Thus, the signal $\gamma(t)$ given in Figure 3 and Figure 4, for example, represents a time varying measure of the average output power at a time τ according to the scaling law

$$\langle \ln P(t) \rangle_\tau = A + \frac{B}{\gamma(\tau)}$$

where $\langle \ln P(t) \rangle_\tau$ denotes the (moving) average value of $\ln P(t)$ at a time τ and A and B are scaling constants associated with a given wind turbine obtained by calibration.

VI. ENERGY QUALITY ESTIMATION FOR WAVE POWER GENERATION

From equation (5), the force generated for a unit mass is given by

$$f(t) = d_t v(t) = \frac{1}{t^{1-1/\gamma}} \otimes_t d_t n(t)$$

Working in a one-dimensional space, the wave equation is then given by (for unit wave speed)

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) u(x,t) = \delta(x) f(t)$$
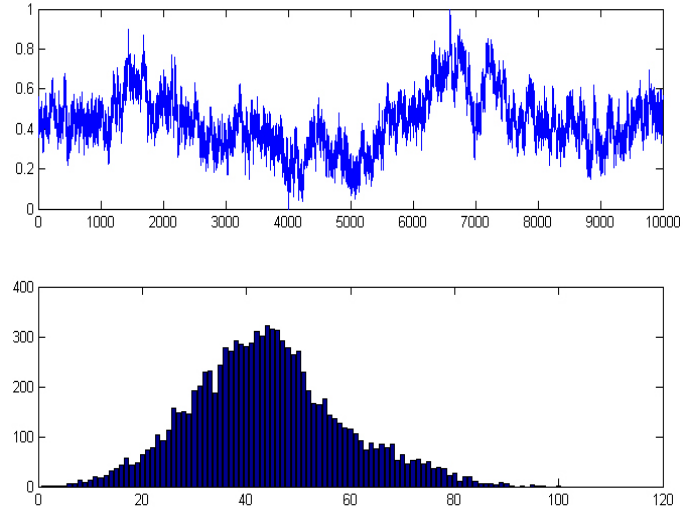

Figure 5. Simulated normalized wind velocities computed for a Lévy index $\gamma = 1.5$ (above) and the corresponding 100-bine histogram (below)
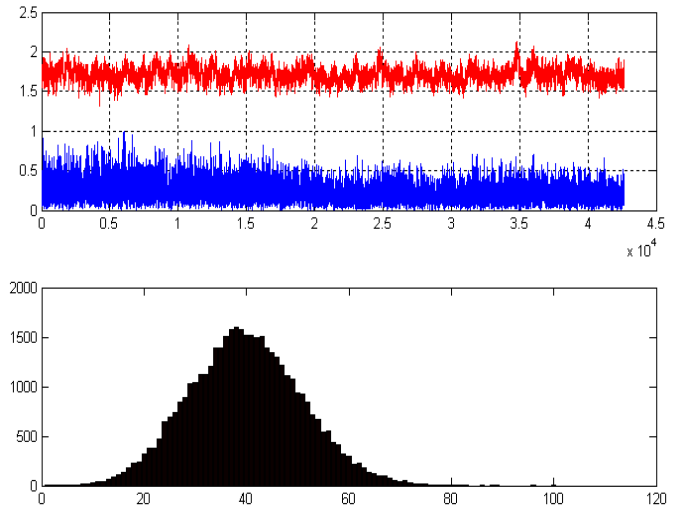

Figure 6. Example of urban data analysis using wind velocities recorded at Dublin Institute of Technology, Kevin Street, Dublin 8 from 14 September 2010 at 22:20:44 to 15 September 2010 at 10:11:51 and sampled in seconds. Above: Normalized wind velocity data $v(t)$ (blue) and the Lévy index $\gamma(t)$ (red) for a look-back moving window of 1024 elements. Below: 100-bin histogram of $\gamma(t)$.

where we considered a source function with a spatial impulse $\delta(x)$. The Green's function solution to this equation is given by (ignoring scaling constants)

$$u(t) = \frac{1}{\pi} \frac{\sin(\Omega t)}{\Omega t} \otimes_t v(t), \quad \Omega \rightarrow 0 \quad \forall x$$

where Ω is the bandwidth of the wave spectrum. The PSDF of $u(t)$ is therefore given by (for scaling constant c)

$$P(\omega) = \mid U(\omega) \mid^2 = \frac{c}{\mid \omega \mid^{2/\gamma}}, \quad \mid \omega \mid \leq \Omega$$

Table III
STATISTICAL PARAMETERS ASSOCIATED WITH THE LÉVY INDEX
FUNCTION GIVEN IN FIGURE 6.

Statistical Parameter	Value for $\gamma(t)$
Minimum Value	1.3209
Maximum value	2.1358
Range	0.8149
Mean	1.7236
Median	1.7204
Standard Deviation	0.0944
Variance	0.0089
Skewness	0.1939
Kertosis	3.0374
Composite Normality	'Reject'

and for a fixed bandwidth Ω, it is clear that the power output depends upon γ associated with the wind velocity according to the model compounded in equation (5). Thus we can consider a time dependent wave power scaling relationship of the form

$$\langle \ln P(\omega) \rangle_\tau = A - \frac{B}{\gamma(\tau)}$$

where A and B are scaling constants for a given wave energy converter determined by calibration.

VII. SUMMARY

We have considered a Lévy distributed model and constructed a fractional diffusion equation for the wind velocity whose temporal solution is characterized by the Lévy index. Analysis of wind velocity data (some examples of which have been provided in this paper) according to this model shows that the Lévy index is a time varying non-Gaussian stochastic function. Based on the data analyzed to date, the index appears to be larger ~ 1.7 for urban areas compared to rural areas when $\gamma \sim 1.5$. These results are consistent with the underlying rationale associated with the model, where, as $\gamma \rightarrow 2$, the stochastic processes become increasingly diffusive. The model presented allows times series for wind velocity to be simulated whose statistical properties are consistent with experimental data (e.g., Figure 5. Moreover, based on the calculations performed in Sections 5 and 6, the Lévy index may provide a useful measure on the power quality of wind turbines and wave energy generators respectively. Further investigation are required to ascertain whether it may be possible to use the signal $\gamma(t)$ for short term predictive analysis on power quality following methods developed for financial risk management [9].

ACKNOWLEDGMENTS

REFERENCES

[1] A. D. Hansen, P. Sorensen, L. Janosi and J. Bech, *Wind Farm Modelling for Power Quality*, Industrial Electronics Society, IECON '01, 27th Annual Conference of the IEEE, Vol. 3, pp. 1959 - 1964, 2001.

[2] A. D. Hansen1, P. Srensen1, F. Blaabjerg and J. Becho, *Dynamic Modelling of Wind Farm Grid Interaction* Wind Engineering, Vol. 26, No. 4, pp. 191-208, 2002.

[3] A. Betz, *Introduction to the Theory of Flow Machines*, Pergamon Press, Oxford, 1966.

[4] A. N. Gorban, A. M. Gorlov and V. M. Silantyev, *Limits of the Turbine Efficiency for Free Fluid Flow*, Journal of Energy Resources Technology, Volume 123, Issue 4, pp. 311-317, 2001.

[5] J. Gottschall and J. Peinke, *Stochastic Modelling of a Wind Turbines Power Output with Special Respect to Turbulent Dynamics*, Journal of Physics: Conference Series 75, The Science of Making Torque from Wind, IOP Publishing, 2007.

[6] M. F. Shlesinger, G. M. Zaslavsky and U. Frisch (Eds.), *Lévy Flights and Related Topics in Physics*, Springer 1994.

[7] S. Abea and S. Thurnerb, *Anomalous Diffusion in View of Einsteins 1905 Theory of Brownian Motion*, *Physica*, A(356), Elsevier, pp. 403-407, 2005.

[8] G. Evans, J Blackledge and P. Yardley, *Analytical Methods for Partial Differential Equations*, Springer, 1999.

[9] J. M. Blackledge, *Application of the Fractional Diffusion Equation for Predicting Market Behaviour*, IAENG International Journal of Applied Mathematics, Vol. 40, Issue 3, pp. 130 - 158, 2010.