Chan-Vese Model with Semi-Implicite AOS Scheme for Images Segmentation: Biphase and Multiphase Cases

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***Abstract*—Active contour models are designed to evolve an initial curve, called level set, to extract the desired object(s) in an image. Various models are used, such as Chan-Vese (CV) model. The CV model has the global segmentation property to segment all objects in an image. The problem with this model is the high time computing. In order to reduce it, our contribution in this work is the association of a semi-implicit Additive Operator Splitting (AOS) technique with the CV model in biphase and multiphase cases. In this paper, we present the new association in biphase and multiphase cases with simulations showing the efficiency of the proposed method.**

Keywords–Image segmentation; actives contours; Chan Vese; AOS scheme.

I. INTRODUCTION

Image segmentation is the task of partitioning an image into multiple regions. The most known region based method has been proposed by Mumford Shah [1] who have introduced a general optimization framework. To determine desired curves or surfaces, this method uses an energy functional based on regional geometric properties such as the area of the region, its contour length and the variation of individual pixel intensities inside and outside the region. However, the Mumford Shah [2] model cannot be easily implemented. The CV method [2] is a special implementation of Mumford Shah using a level set function for the case of two phases with two piecewise constants. The basic idea of CV model is to minimize energy functional by solving the Euler-Lagrange equation. This minimisation takes enough time in image segmentation.

To reduce the time of segmentation, Weickert et al. [3] provide a fast algorithm using the semi-implicit AOS scheme. The basic idea behind the AOS schemes is to decompose a multi-dimensional problem into one-dimensional ones that can be solved very efficiently. Then the final multi-dimensional solution is approximated by averaging the one-dimensional solutions. In [4], the authors present a combination of the semi-implicite AOS scheme and a narrow-band technique which is associated to the geodesic active contours. This association requires re-initialization for each iteration which is the weakness of the method. As solution, Kuhne et al. [5] provide a fast algorithm using an semi-implicit AOS scheme technique which is suitable both for the geometric and the geodesic active contour model. In [6], the authors propose a new selective segmentation model, combining ideas from global segmentation, that can be reformulated in a convex way such that a global minimizer can be found independently of initialization. They present the Convex Distance Selective Segmentation (CDSS) functional (based on CV model) which is associated with the semi-implicite AOS scheme. In our work, we use a level set representation of the CV model with the semi-implicite AOS scheme in order to improve the speed of the segmentation in biphase and multiphase cases.

This paper is organized as follows. Section 2 contains a review of level set method and the CV model for biphase and multiphase cases. In Section 3, we present the semi-implicite AOS scheme. Then, we present the CV model with the semi-implicite AOS scheme in biphase and multiphase cases in Section 4. Experimental results are given in Section 5.

II. ACTIVE CONTOUR MODELS

In this section, we shall first provide an overview of level set theory before we get into the details of the CV model.

A. Level set method

A level set method is a numerical technique, which helps with tracking moving fronts to interfaces and shapes. This technique was first introduced by Osher et al. in [7], where the boundaries are given by level sets of a function $\phi(\mathbf{x})$, naming it as the level set method. This method is very successful due to a very easy way of following shapes that change topology. For a given interface $\Gamma = \partial\Omega$ as shown in Figure 1, the level set is independent of the parametrisation of the contour and can be used to represent the interface evolution. The idea of the level set method is to implicitly represent an interface Γ as the level set of a function ϕ. The level set function ϕ of the closed front Γ is defined as follows:

$$\begin{cases} \phi(\mathbf{x}) > 0 \ inside\ \Gamma \\ \phi(\mathbf{x}) < 0 \ outside\ \Gamma, \\ \phi(\mathbf{x}) = 0 \ on\ \Gamma. \end{cases}$$

Where $\mathbf{x} \in R^2$.

The adjusting contour at time t is denoted by $\phi(\mathbf{x}(t);t)$

$$\begin{cases} \phi(\mathbf{x}(t);t) > 0 \ inside\ \Gamma \\ \phi(\mathbf{x}(t);t) < 0 \ outside\ \Gamma, \\ \phi(\mathbf{x}(t);t) = 0 \ on\ \Gamma. \end{cases}$$

The level set value of a point on the contour with motion must always be 0.

$$\phi(\mathbf{x}(t);t) = 0 \tag{1}$$

A derivation of (1) with respect to t and after some manipulation, yields PDE equation:

$$\frac{\partial\phi}{\partial t} + F|\nabla\phi| = 0 \tag{2}$$

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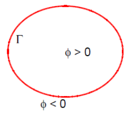

Figure 1. Representation of the interface Γ.

Where F stands for the speed in which the contour propagates in normal direction with an initial condition $\phi(\mathbf{x}, t = 0)$ (the initial drawn curve).

B. The CV model

1) biphase case: In [2], the authors present a special implementation of the CV method based on the use of the level set method to minimize the piecewise constant two phases Mumford Shah functional [1]. The advantage of this implementation is the possibility to detect objects whose boundaries are not necessarily defined by gradient and overcame the problematic tracking of Γ. For a given image u_0 in domain Ω, the CV model is formulated by minimizing the following energy functional :

$$\begin{aligned} F^{CV} &= \mu \int_\Omega \delta(\phi)|\nabla\phi| dxdy + \nu \int_\Omega H(\phi) dxdy + \\ &\lambda_1 \int_\Omega |u_0(x,y) - c_1|^2 H(\phi(x,y)) dxdy + \\ &\lambda_2 \int_\Omega |u_0(x,y) - c_2|^2 (1 - H(\phi(x,y))) dxdy \end{aligned} \tag{3}$$

Where μ, λ_1 and λ_2 are positive parameters, ϕ is a level set function, $H(\phi)$ is the Heaviside function and $\delta(\phi)$ is the Dirac function. Generally, the regularized versions are selected as follows:

$$\begin{cases} H_\epsilon(\phi) = \frac{1}{2}\left(1 + \frac{2}{\pi}\arctan\left(\frac{\phi}{\epsilon}\right)\right), \\ \delta_\epsilon(\phi) = \frac{1}{\pi}\frac{\epsilon}{\phi^2+\epsilon^2}. \end{cases} \tag{4}$$

The two piecewise constants c_1 and c_2 are defined as

$$c_1 = \frac{\int_\Omega u_0(x,y) H_\epsilon(\phi(x,y)) dxdy}{\int_\Omega H_\epsilon(\phi(x,y)) dxdy}, \tag{5}$$

$$c_2 = \frac{\int_\Omega u_0(x,y)(1 - H_\epsilon(\phi(x,y))) dxdy}{\int_\Omega (1 - H_\epsilon(\phi(x,y))) dxdy}, \tag{6}$$

The evolution equation is given by :

$$\frac{\partial\phi}{\partial t} = \delta_\epsilon(\phi)[\mu\nabla.(\frac{\nabla\phi}{|\nabla\phi|}) - \nu - \lambda_1(u_0 - c_1)^2 + \lambda_2(u_0 - c_2)^2] \tag{7}$$

2) multiphase case: The CV model for multiphase piecewise constant (we use two level set functions ϕ_1 and ϕ_2) is formulated by minimizing the following energy functional [8]:

$$\begin{aligned} F_{\epsilon,4} &= \int_\Omega (u_0 - c_{11})^2 H_\epsilon(\phi_1) H_\epsilon(\phi_2) dxdy \\ &+ \int_\Omega (u_0 - c_{10})^2 H_\epsilon(\phi_1)(1 - H_\epsilon(\phi_2)) dxdy \\ &+ \int_\Omega (u_0 - c_{01})^2 (1 - H_\epsilon(\phi_1)) H_\epsilon(\phi_2) dxdy \\ &+ \int_\Omega (u_0 - c_{00})^2 (1 - H_\epsilon(\phi_1))(1 - H_\epsilon(\phi_2)) dxdy \\ &+ \mu \int_\Omega |\nabla H_\epsilon(\phi_1)| + \mu \int_\Omega |\nabla H_\epsilon(\phi_2)| \end{aligned} \tag{8}$$

Where

$$c_{11}(\phi) = \frac{\int_\Omega u_0 H_\epsilon(\phi_1) H_\epsilon(\phi_2) dxdy}{\int_\Omega H_\epsilon(\phi_1) H_\epsilon(\phi_2) dxdy}, \tag{9}$$

$$c_{10}(\phi) = \frac{\int_\Omega u_0 H_\epsilon(\phi_1)(1 - H_\epsilon(\phi_2)) dxdy}{\int_\Omega H_\epsilon(\phi_1)(1 - H_\epsilon(\phi_2)) dxdy}, \tag{10}$$

$$c_{01}(\phi) = \frac{\int_\Omega u_0 (1 - H_\epsilon(\phi_1)) H_\epsilon(\phi_2) dxdy}{\int_\Omega (1 - H_\epsilon(\phi_1)) H_\epsilon(\phi_2) dxdy}, \tag{11}$$

$$c_{00}(\phi) = \frac{\int_\Omega u_0 (1 - H_\epsilon(\phi_1))(1 - H_\epsilon(\phi_2)) dxdy}{\int_\Omega (1 - H_\epsilon(\phi_1))(1 - H_\epsilon(\phi_2)) dxdy}, \tag{12}$$

Evolution equations of ϕ_1 and ϕ_2 are given by:

$$\begin{aligned} \frac{\partial\phi_1}{\partial t} &= \delta_\epsilon(\phi_1)\{\mu div(\frac{\nabla\phi_1}{|\nabla\phi_1|}) \\ &-[((u_0 - c_{11})^2 - (u_0 - c_{01})^2)(H_\epsilon(\phi_2)) \\ &+((u_0 - c_{10})^2 - (u_0 - c_{00})^2)(1 - H_\epsilon(\phi_2))]\} \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{\partial\phi_2}{\partial t} &= \delta_\epsilon(\phi_2)\{\mu div(\frac{\nabla\phi_2}{|\nabla\phi_2|}) \\ &-[((u_0 - c_{11})^2 - (u_0 - c_{10})^2)(H_\epsilon(\phi_1)) \\ &+((u_0 - c_{01})^2 - (u_0 - c_{00})^2)(1 - H_\epsilon(\phi_1))]\} \end{aligned} \tag{14}$$

III. AOS SCHEME

The AOS method is proposed by Tai et al. in [9] and Weickert et al. in [3]. The AOS scheme guarantees equal treatment of all coordinate axes and is stable for big time steps. The scheme presents the semi-implicit algorithm based on a discrete non-linear diffusion scale-space framework. This scheme is applied to the m-dimensional diffusion equation and it is given in the following form:

$$\frac{\partial\phi}{\partial t} = div(g\nabla\phi) + f(\mathbf{x}, \phi). \tag{15}$$

$$\frac{\partial\phi}{\partial t} = \sum_{j=1}^{m} \frac{\partial}{\partial x_j}(g_j(\phi)\frac{\partial\phi}{\partial x_j}) + f(\mathbf{x}, \phi). \tag{16}$$

Where $[0, T] \times \Omega \subset \mathbb{R}^m$. The initial and boundary conditions are:

$$\phi(0, .) = \phi_0 \ \ and \ \ \frac{\partial\phi}{\partial n} = 0 \ \ on \ \partial\Omega,$$

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We consider discrete times $t_k = k\Delta t$, where $k \in \mathbb{N}_0$ and Δt a semi-implicit discretization of the diffusion equation.

$$\phi^{k+1} = \left(I - \Delta t \sum_{l=1}^{m} A_l(\phi) \right)^{-1} \hat{\phi}^k, \; k = 1, 2, ... \qquad (17)$$

Where $\hat{\phi}^k = \phi^k + \Delta t f$.

We may consider AOS variant (for m=2)

$$\phi^{k+1} = \frac{1}{2} \sum_{l=1}^{2} \left(I - 2\Delta t A_l(\phi^k) \right)^{-1} \hat{\phi}^k, \; k = 1, 2, ... \qquad (18)$$

The AOS scheme offers one important advantage [10] : the operators $B_l(u^k) = I - 2\Delta t A_l(\phi^k)$ lead to strictly diagonally dominant tridiagonal linear systems, which can be solved very efficiently with Thomas algorithm. This algorithm has a linear complexity and can be implemented very easily.

To implement equation (18), we proceed in three steps [10]:

1) Evolution in x direction with step size $2\Delta t$: Solve the tridiagonal system $\left(I - 2\Delta t A_x(\phi^k) \right) \upsilon^{k+1} = \hat{\phi}^k$ for υ^{k+1}.
2) Evolution in y direction with step size $2\Delta t$: Solve the tridiagonal system $\left(I - 2\Delta t A_y(\phi^k) \right) \omega^{k+1} = \hat{\phi}^k$ for ω^{k+1}.
3) Averaging: Compute $\phi^{k+1} := 0.5(\upsilon^{k+1} + \omega^{k+1})$.

IV. THE CV MODEL WITH THE SEMI-IMPLICITE AOS SCHEME

In this section, we present the CV model with the semi-implicite AOS scheme in biphase and multiphase cases.

A. Biphase case

From equation (7), we denote:

$$\begin{aligned} f &= \delta_\epsilon(\phi)\{-[\lambda_1(u_0 - c_1)^2 - \lambda_2 \\ &\quad (u_0 - c_2)^2] - \nu\}. \end{aligned} \qquad (19)$$

To avoid singularities, we replace the term $|\nabla\phi|$ with $|\nabla\phi|_\beta = \sqrt{\phi_x^2 + \phi_y^2 + \beta}$ and denote $W = frac1|\nabla\phi|_\beta$.

Discretizing (7) by employing the AOS scheme, we get the following equation:

$$\phi^{n+1} = \frac{1}{2} \sum_{l=1} 2(I - 2\Delta t A_l(\phi^n))^{-1} \hat{\phi}^n \qquad (20)$$

The matrices A_l, for $l = 1, 2$, are tridiagonal matrices derived using finite differences [11] and $\hat{\phi}^n = \phi^n + \Delta t f$. One modification is introduced on the AOS equation is in A_1 and A_1, where we add the term $\mu\delta_\epsilon(\phi^n)$ because we work directly with the level set function ϕ.

$$\begin{aligned} (A_1(\phi^n)\phi^{n+1})_{i,j} &= \mu\delta_\epsilon(\phi^n)\frac{E^n_{i+1,j} + E^n_{i,j}}{2h_x^2} \\ &\quad (\phi^{n+1}_{i+1,j} - \phi^{n+1}_{i,j}) \\ &\quad -\mu\delta_\epsilon(\phi^n)\frac{E^n_{i,j} + E^n_{i-1,j}}{2h_x^2} \\ &\quad (\phi^{n+1}_{i,j} - \phi^{n+1}_{i-1,j}) \end{aligned}$$

$$\begin{aligned} (A_2(\phi^n)\phi^{n+1})_{i,j} &= \mu\delta_\epsilon(\phi^n)\frac{E^n_{i,j+1} + E^n_{i,j}}{2h_y^2} \\ &\quad (\phi^{n+1}_{i,j+1} - \phi^{n+1}_{i,j}) \\ &\quad -\mu\delta_\epsilon(\phi^n)\frac{E^n_{i,j} + E^n_{i,j-1}}{2h_y^2} \\ &\quad (\phi^{n+1}_{i,j} - \phi^{n+1}_{i,j-1}) \end{aligned}$$

The algorithm of the CV model with the semi-implicit AOS in biphase case is:

1) Initialize ϕ^0 by ϕ_0, k=0.
2) compute f from equation (19),
3) Compute $c_1(\phi^k)$ and $c_2(\phi^k)$ by (5) and (6).
4) Compute $\phi^{(k)}$ using (20).
5) Check whether the solution is stationary. If not, repeat 2-5

B. multiphase case

From equation (13), we denote :

$$\begin{aligned} f_1 &= \delta_\epsilon(\phi_1)\{-[((u_0 - c_{11})^2 - (u_0 - c_{01})^2) \\ &\quad (H_\epsilon(\phi_2)) + ((u_0 - c_{10})^2 \\ &\quad -(u_0 - c_{00})^2)(1 - H_\epsilon(\phi_2))]\} \end{aligned} \qquad (21)$$

From equation (14), we denote :

$$\begin{aligned} f_2 &= \delta_\epsilon(\phi_2)\{-[((u_0 - c_{11})^2 - (u_0 - c_{10})^2) \\ &\quad (H_\epsilon(\phi_1)) + ((u_0 - c_{01})^2 \\ &\quad -(u_0 - c_{00})^2)(1 - H_\epsilon(\phi_1))]\} \end{aligned} \qquad (22)$$

To avoid singularities, we replace the term $|\nabla\phi_1|$ with $|\nabla\phi_1|_\beta = \sqrt{\phi_{1x}^2 + \phi_{1y}^2 + \beta}$ and $|\nabla\phi_2|$ with $|\nabla\phi_2|_\beta = \sqrt{\phi_{2x}^2 + \phi_{2y}^2 + \beta}$

The algorithm of the CV model with the semi-implicit AOS in multiphase case is:

1) Initialize ϕ_1^0 and ϕ_2^0 by ϕ_{1_0} and ϕ_{2_0} , k=0.
2) compute $c_{11}(\phi^k)$, $c_{10}(\phi^k)$, $c_{01}(\phi^k)$ et $c_{00}(\phi^k)$
3) compute f_1 and f_2 by equation (21) and (22).
4) Compute $\phi_1^{(k)}$ using (20) and $\phi_2^{(k)}$ using (20)
5) Check whether the solution is stationary. If not, repeat 2-5

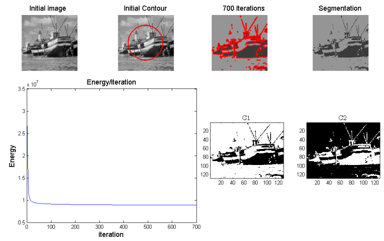

Figure 2. Segmentation by CV model (biphase case) of boat.

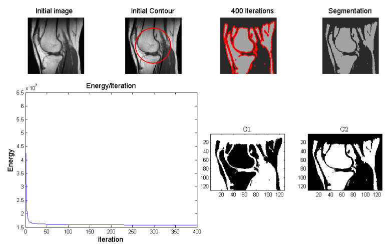

Figure 3. Segmentation by CV model (biphase case) of MR image of knee.

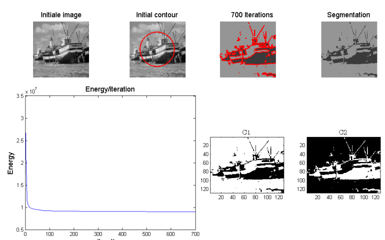

Figure 4. Segmentation by the CV model with semi-implicite AOS scheme (biphase case) of boat.

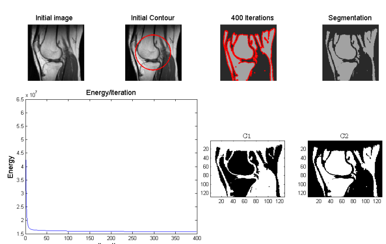

Figure 5. Segmentation by the CV model with semi-implicite AOS scheme (biphase case) of MR image of knee.

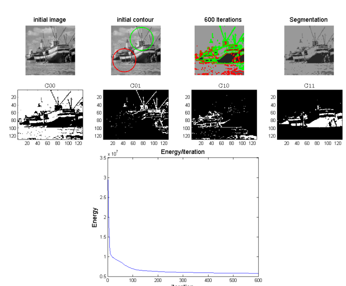

Figure 6. Segmentation by CV model (multiphase case) of boat.

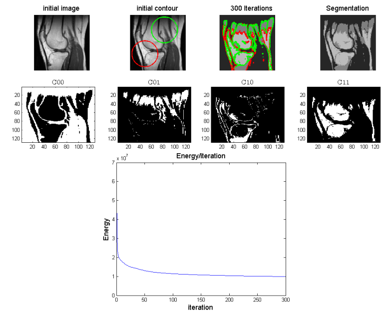

Figure 7. Segmentation by CV model (multiphase case) of MR image of knee.

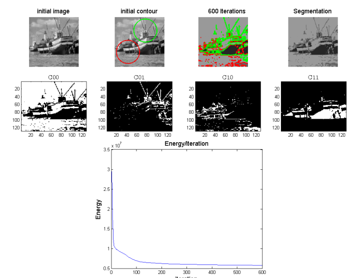

Figure 8. Segmentation by the CV model with semi-implicite AOS scheme (multiphase case) of boat.

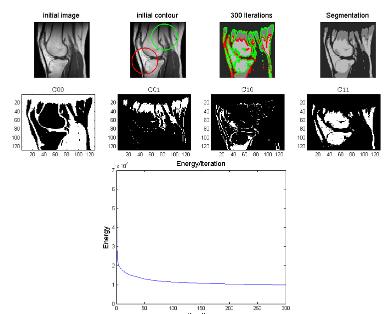

Figure 9. Segmentation by the CV model with semi-implicite AOS scheme (multiphase case) of MR image of knee.

V. EXPERIMENTAL RESULTS

In the biphase case, the constants are given as follow $\nu = 0$, $\Delta t = 1$ and $\lambda_1 = \lambda_2 = 1$. In Figures 2 and 3, we illustrate the segmentation by the CV model for boat and MR of knee images. In Figures 4 and 5, we show the segmentation by the CV model with semi-implicite AOS scheme for the same images. The segmentation illustrates the two phases and the results are almost similar for the two methods. For the multiphase case, the constants are given as follow $\nu = 0$ and $\lambda_1 = \lambda_2 = 1$. In Figures 6 and 7, we illustrate the segmentation by the CV model for boat and MR of knee images, but in Figures 8 and 9 we show the segmentation by the CV model with the semi-implicite AOS scheme for the same images. The two methods give exactly the same segmentation where we can see the four phases. The comparison study relative to time computing is summarized in Tables I and II; we deduce that the CV model with semi-implicite AOS scheme reduces the time computing of the segmentation by half.

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TABLE I. COMPARISON BETWEEN THE CV MODEL AND THE CV MODEL WITH THE SEMI-IMPLICITE AOS SCHEME IN BIPHASE CASE.

Image	**Boat**		**MR image of knee**	
Method	*CV*	CV-AOS	*CV*	CV-AOS
CPU time (s)	110.6671	56.7532 s	51.9639	22.1521

TABLE II. COMPARISON BETWEEN THE CV MODEL AND THE CV MODEL WITH THE SEMI-IMPLICITE AOS SCHEME IN MULTIPHASE CASE.

Image	**Boat**		**MR image of knee**	
Method	*CV*	CV-AOS	*CV*	CV-AOS
CPU time (s)	158.2630	71.0429	70.3253 s	28.1270

VI. CONCLUSION

In this paper, we have used the advantages of the semi-implicit AOS technique in order to fast the CV model for image segmentation in biphase and multiphase cases. The experimental results show that the segmentation is done in the two cases, with the the superiority of the CV model with the semi-implicite scheme compared to the CV model concerning the time computing. As future work, we plan to associate the semi-implicit AOS technique with other active contour models.

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