Principal Fiber Bundles and Geometry of Color Spaces

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***Abstract*—In 1974, H.L.Resnkikoff published an inspiring paper about the use of differential geometry to study, among others, the intrinsic shape of the space of perceived colors and the Riemannian metrics on it. The mathematical techniques that he used is shared with modern theories of theoretical physics, which are far from being a common background for scientists in color vision and processing. Due to this, Resnikoff's paper remained unnoticed for decades. In this brief contribution, some insights about how to update Resnikoff's ideas will be given and discussed in relationship with a moder theory of color spaces and to the mathematical concept of principal fiber bundle.**

Keywords–Geometry of Color Spaces; Principal Fiber Bundles; Human Perception; Applications to Perceived Color Distances.

I. Resnikoff's framework for the space of perceived colors

In the 1974 paper [1], H.L. Resnikoff analyzed the geometrical and topological properties of the space of perceived colors $\mathcal{P}$ with a high level mathematical rigor. He decided to start from Schrödinger's axioms [2] for $\mathcal{P}$: *Axiom 1* (Newton 1704): if $x \in \mathcal{P}$ and $\alpha \in \mathbb{R}^+$, then $\alpha x \in \mathcal{P}$. *Axiom 2*: if $x \in \mathcal{P}$ then it does not exist any $y \in \mathcal{P}$ such that $x + y = 0$. *Axiom 3* (Grassmann 1853, Helmholtz 1866): for every $x, y \in \mathcal{P}$ and for every $\alpha \in [0, 1]$, $\alpha x + (1 - \alpha)y \in \mathcal{P}$. *Axiom 4* (Grassmann 1853): every collection of more than three perceived colors is a linear dependent family in the vector space V spanned by the elements of $\mathcal{P}$. Note, in particular, that Axiom 3 implies that $\mathcal{P}$ is closed under convex linear combinations, i.e., every two colors in $\mathcal{P}$ can be joined by a line segment, i.e., $\mathcal{P}$ is *convex*.

Resnikoff added another axiom, that of *local homogeneity* of $\mathcal{P}$ with respect to changes of background illumination of the visual scene. If X is a topological space and G is a group of transformations that acts on X, then X is called a *homogeneous space with respect to* G if, for any two points $x, y \in X$, there exists a transformation $g \in G$ such that $g(x) = y$, i.e., any two points of X can be joined by an opportune transformation g induced by G. X is only *locally homogeneous with respect to* G if this property holds only locally, i.e., if for every $x \in X$ there is an open neighborhood U_x containing it and such that every $x' \in U_x$ can be written as $x' = g(x)$ for a certain $g \in G$. The reason for introducing this further axiom is that it is possible to modify a color to reach a 'very similar' color with a change of illumination and this means that $\mathcal{P}$ should be locally homogeneous with respect to the group of transformations of illuminations.

Resnikoff claimed that this group can be assumed as the following:

$$GL(\mathcal{P}) := \{g \in GL(V) \mid g(x) \in \mathcal{P} \ \ \forall x \in \mathcal{P}\},$$

where $GL(V)$ is the group of orientation-preserving invertible linear operators on V, or, equivalently, the group of real $n \times n$ matrices with determinant greater than zero, where $n = dim(V) \leq 3$ thanks to Axiom 4. He justifies this choice from the consideration that Axiom 1 implies that $\mathcal{P}$ is a cone embedded in V and so a general transformation of illumination must preserve the orientation of the cone and it must also be invertible, since it is possible to turn back to the initial conditions of illuminations. The condition $g(x) \in \mathcal{P}$ is perfectly natural because after the change of illumination we can still perceive the colors.

The observation that a change of illumination slightly modifies the perception of colors of a visual scene can thus be stated in this mathematical formalism by saying that $\mathcal{P}$ is locally homogeneous with respect to $GL(\mathcal{P})$. But, thanks to Axiom 3, for every couple of perceived colors $x, y \in \mathcal{P}$ there exists the line segment that join x to y. This segment is compact, hence it can be covered by a finite partition of open neighborhoods $U_1, \ldots, U_n$ and the color x can be moved along this line segment passing from a neighborhood to the next one with the transformations $g_1, \ldots, g_n$. Thus, the global transformation that enables us to pass from x to y is the composition of the single transformations, i.e., $y = g(x)$, $g = g_n \circ \cdots \circ g_1$ and so local homogeneity for the convex $\mathcal{P}$ implies its global homogeneity.

For this reason, Resnikoff postulates a fifth axiom on the structure of the color space: *Axiom 5* (Resnikoff 1974): $\mathcal{P}$ is globally homogeneous with respect to the group of transformations of illumination $GL(\mathcal{P})$.

Starting from the set of axioms 1-5 and by using Lie groups and algebras representation theory [3], Resnikoff managed showed that the only two geometrical structures compatible with these axioms are:

$$\mathcal{P} \simeq \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,$$

or

$$\mathcal{P} \simeq \mathbb{R}^+ \times SL(2, \mathbb{R})/SO(2),$$

where $SL(2, \mathbb{R})$ is the group of 2×2 matrices with real entries and determinant $+1$ and $SO(2)$ is the group of matrices that perform rotations in the plane $\mathbb{R}^2$.

The first geometrical structure agrees with the usual trichromatic space, such as RGB, XYZ, and so on. The Cartesian product, which represents the second one, is totally new.

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Moreover, he found out that the only color constancy metric compatible with the homogeneous structure of the RGB-like model is precisely the metric found by Stiles, i.e.,

$$ds^2 = \alpha_1 \left(\frac{dx_1}{x_1}\right)^2 + \alpha_2 \left(\frac{dx_2}{x_2}\right)^2 + \alpha_3 \left(\frac{dx_3}{x_3}\right)^2,$$

where $x_j \in \mathbb{R}^+$ and α_j are positive real constants for $j = 1, 2, 3$. This metric agrees with the one found by Stiles with totally different methods.

The distinction of perception between the color x and the color y is calculated with the integral

$$d(x, y) = \int_\gamma ds \qquad \gamma(0) = x,\, \gamma(1) = y,$$

where γ is the unique geodesic arc (i.e the arc of minimal length) between x and y. For more information, the interested reader is referred to [4].

II. Updating Resnikoff's model: Principal fiber bundles

Resnikoff's model is one of the most elegant treatises on color perception and it paved the road to the introduction of some advanced mathematical techniques used in theoretical physics, e.g., differential geometry, Lie groups and algebras representation theory and Jordan algebras, to the theory of color perception. In this section, it is discussed the idea that another fundamental mathematical object commonly used in classical and quantum field theory of mathematical physics, the principal fiber bundle, can be a fundamental (missing) piece in the Resnikoff framework.

First of all, note that Axiom 1 fails for $\alpha \simeq 0$ and $\alpha \gg 1$. In fact, as α approaches zero, the retinal cones responsible for color vision do not work anymore and retinal rods are activated, allowing only black and white vision, which can be identified with achromatic colors in $\mathcal{P}$. However, rods sensitivity is finite, so that under a certain threshold $\bar{\alpha}$, vision ceases and with it the geometric structure of $\mathcal{P}$. The same can be said when α overcomes an upper limit, after which retinal cones saturate and sight is lost.

A second issue is that, in Resnikoff's model, only independent light stimuli over a uniform background are considered; however, color vision in real world conditions is much more complex. In fact, color perception of natural scenes is intrinsically local: hue, saturation and brightness of a patch strongly depend on the surrounding patches, a phenomenon called 'induction'. This is the reason why one must distinguish between spectral colors of light sources isolated from the rest of the visual field, and *color in context*. Induction analysis is an active research field both in image processing and cognitive psychology, see e.g., [5]–[9].

When induction phenomena are taken into account, it is clear that if we want to represent color differences a spatially variant Riemannian metric on $\mathcal{P}$ must be considered, instead of a global one. This is where the framework of principal fiber bundles [10] [11] can be helpful.

Without entering in the very complicated matter of field theory, it is nevertheless possible to give an idea of what fiber bundles are by considering a *field* as an entity which assigns to every point x of a manifold M a point f of another manifold F, representing the value taken by the field in x. A *configuration* of a field on an open subset U of M is a map $\varphi : U \subset M \to F$ completely defined by its graph, i.e., by the set $\mathrm{Graph}(\varphi) := \{(x, f) \in U \times F \mid f = \varphi(x)\}$.

It is quite natural to think at $U \times F$ as the local model of a more complicated geometric structure obtained by 'gluing together' these Cartesian products (in a suitable way). This structure is precisely what is called a fiber bundle over M with standard fiber F. Hence, naively, a fiber bundle can be seen as a generalization of the concept of a manifold, now modeled on a Cartesian product instead of an Euclidean space. A fiber bundle is a *principal bundle* if the standard fiber is a Lie group G.

The importance of considering Lie groups has been discussed in the previous section, thus it seems necessary to consider, among all fiber bundles, principal fiber bundles as the candidates to provide the rich geometrical structure needed to introduce in Resnikoff's framework the phenomenon of local induction. A formalization of this idea can lead to new, context-dependent, color metrics rigorously obtained from first principles and not by ad-hoc procedures.

III. Conclusion

The Resnikoff's model of perceived color space has been recalled and some critics about its assumptions have been pointed out. These observations can be the staring point for a new analysis of color spaces, based on the mathematical concept of principal fiber bundles, which it has been motivated to seem the most adequate framework to further develop Resnikoff's analysis.

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