SL(2,R) Multi-scale Contour Registration Based on Riemannian Calculation

Khaoula Sakrani
CRISTAL Laboratory, GRIFT Group
National School of Computer Science
Manouba university, Tunisia
Email. sakrani.khaoula@gmail.com

Sinda Elghoul
CRISTAL Laboratory, GRIFT Group
National School of Computer Science
Manouba university, Tunisia
Email. sinda.elghoul@ensi-uma.tn

Faouzi Ghorbel
CRISTAL Laboratory, GRIFT Group
National School of Computer Science
Manouba university, Tunisia
Email. faouzi.ghorbel@ensi-uma.tn

***Abstract*—We introduce a novel Affine multi-scale registration based on the Riemannian metric in the Lie group $SL(2, R)$ to estimate the best alignment between two planar curves. First, we smooth and re-simpling the input shapes. Then, in each level, we compute the special linear transformations A_{σ_p} and translation vectors B_{σ_p} using the pseudo-inverse algorithm. The obtained matrices A_{σ_p} are then projected in the Lie algebra of $SL(2, R)$ which is $sl(2, R)$ to compute their average. In the final step, we register and calculate the L_2 distance.**

***Keywords*—*Multi-scale registration; Special affine transformation; Riemannian metric; Affine Spacial group* $SA(2, R)$.**

I. INTRODUCTION

The comparison process between images is complicated and restricted when the images were captured using multiple sensors and poses and were not shot simultaneously. Most of the time, a machine will not be able to find the same thing in different pictures because it can change. In this situation, it is challenging to integrate two comparable forms. To address these issues, researchers created different curve registration methods. The main goal of this method is to find the geometric transformation between two or more images in order to get the most desirable alignment. The registration of the planar curves' shapes is the optimum solution that has been presented for a great number of applications, including motion tracking [1], mosaicing [2] [3], object recognition [4], remote sensing [5], 3D curve reconstruction [6] [7] and medical image analysis [8] [9]. Different methods of shape registration have been proposed in recent years to estimate motion and align two shapes. Thus, 2D affine shapes can be registered using techniques that rely on the Riemannian calculation. The authors in [10] introduce a subspace method for aligning two 2D shapes and estimating the affine transformation between them. By minimizing the projection error in the subspace spanned by the two shapes, the affine transformation is estimated in the proposed 2D signal method. Bryner et al. [11] propose a broad Riemannian framework for shape analysis of planar objects, whose metrics and related quantities are invariant under the action of affine and projective groups. Within the framework of landmark-based shape analysis, Sparr [12] develops affine shape theory through the use of subspace computations. Begelfor and Werman [13] provide a Riemannian geometric metric for computing the averages and distributions of point configurations, such that configurations up to affine transformations are regarded as equivalent. Also, authors in [14] introduce a framework for contour-based shape analysis based on Riemannian geometry that is robust against affine transformation and contour re-parameterization. By integrating the Iwasawa decomposition of $GL(2, R)$ and Lie group parametrization into the regular Iterative Closest Point (ICP) method, Ying et al. [17] introduce new techniques for 2D affine shape registration. Moreover, authors in [18] show how to find a geodesic that is invariant to scale, translation, rotation, and re-parameterization using a Riemannian quasi-Newton approach. YI MA [28] highlights how multiple-view geometry can be studied in three-dimensional spaces with constant curvatures, like Euclidean space, spherical space, and hyperbolic space. In [29], the authors talk about the manifold and Lie group SO(n) of special orthogonal related to the non-negative independent component analysis (ICA). Huang et al. [30] come up with a new way to use Riemannian optimization to align curves in elastic shape analysis.

The purpose of this paper is to introduce a novel Affine Multi-Scale Curve Registration that employs Riemannian geometry. For this technique, two curves are taken as input (the source image and the target image), and then they are sequentially smoothed and reparametrized with affine arc-length. The pseudo-inverse algorithm is then used to compute the special linear transformations A_{σ_p} and translation vectors B_{σ_p} for each smoothed and reparametrized shape. These matrices A_{σ_p} belong to the affine spacial group $SA(2, R)$. The average of these matrices, A, is then found using Riemannian calculation in $SA(2, R)$. Finally, the alignment process is done.

The following is the outline for this paper: In Section II, we present the affine multi-scale curve registration based on the Riemannian calculation that we propose. In Section III, we assess the performance of the suggested methods for shape retrieval with MCD. Ultimately, a final conclusion is reached.

II. AFFINE MULTI-SCALE CURVE REGISTRATION BASED ON RIEMANNIAN CALCULATION

Here, we will talk about the main parts of the proposed method, which is called Affine Multi-Scale Curve Registration based on Riemannian calculation. In this new method, the input normalized contours are filtered over and over again, and the Riemannian calculation in the special linear group SL(2, R) is used to find the best transformation. Fig. 1 demonstrates the Affine Multi-Scale Curve Registration based on the Riemannian calculation procedure.

- A-1: Normalize the input shapes f and h using the affine arc-length normalization [15]. Fig.2 shows two shapes that have been normalized with affine arc-length.
- A-2: Convolve each of the two re-sampling curves using the Gaussian calculation [15], where the resulting curve is depicted in Fig.3.
- A-3: The obtained p systems at each level are formed by the following $2N$ linear equations.

$$\begin{cases} h_{\sigma_1}(l_1) = A_{\sigma_1} f_{\sigma_1}(l_1) + B_{\sigma_1} \\ h_{\sigma_1}(l_2) = A_{\sigma_1} f_{\sigma_1}(l_2) + B_{\sigma_1} \\ \ldots \\ h_{\sigma_1}(l_N) = A_{\sigma_1} f_{\sigma_1}(l_N) + B_{\sigma_1} \end{cases} \begin{cases} h_{\sigma_2}(l_1) = A_{\sigma_2} f_{\sigma_2}(l_1) + B_{\sigma_2} \\ h_{\sigma_2}(l_2) = A_{\sigma_2} f_{\sigma_2}(l_2) + B_{\sigma_2} \\ \ldots \\ h_{\sigma_2}(l_N) = A_{\sigma_2} f_{\sigma_2}(l_N) + B_{\sigma_2} \end{cases} \tag{1}$$

$$\ldots \begin{cases} h_{\sigma_p}(l_1) = A_{\sigma_p} f_{\sigma_p}(l_1) + B_{\sigma_p} \\ h_{\sigma_p}(l_2) = A_{\sigma_p} f_{\sigma_p}(l_2) + \hat{B}_{\sigma_p} \\ \ldots \\ h_{\sigma_p}(l_N) = A_{\sigma_p} f_{\sigma_p}(l_N) + B_{\sigma_p} \end{cases}$$

- A-4, A-5: The A_{σ_p} matrices, which contain the elements of the special affine group $SA(2, R)$, and the B_{σ_p} translation vectors, are obtained by performing the pseudo-inverse calculation [16] on each system.
- A-6: **Riemannian calculation in $SA(2, R)$.**
  We provide a brief introduction to the Special Affine $SA(2)$, which is the underlying geometric space for non-rigid registration. In affine space, the special affine group consists of transformations by scaling, rotation, and then translation. Specifically, it is the semi-direct product of the Special Linear group $SL(2)$ and R^2.

$$SA(2) = SL(2) \times R^2 \tag{2}$$

  It is worth remembering that a Lie group is both a group and a differential manifold, and that a Lie algebra is a vector space on which a Lie bracket is defined.

  The $SL(2, R)$ special linear group contains all determinants of unit size that are real matrices of size 2 by 2.

$$SL(2, R) = \left\{A \in R^2 / det(A) = 1\right\} \tag{3}$$

  An Iwasawa decomposition exists for this 2-dimensional Lie group $SL(2, R)$ of real matrices.

$$SL(2, R) = A_{shear} A_{scale} A_{rotation} \tag{4}$$

  In our case, the affine transformation matrices $A_{\sigma_p} \in SL(2, R)$ and A_{shear} represent shears, A_{scale} is for scales and $A_{rotation}$ list the rotation matrices.

$$A_{\sigma_p} = \begin{pmatrix} a_{11_{\sigma_p}} & a_{12_{\sigma_p}} \\ a_{21_{\sigma_p}} & a_{22_{\sigma_p}} \end{pmatrix}$$

$$A_{\sigma_p} = A_{shear} A_{scale} A_{rotation} \tag{5}$$

$$A_{\sigma_p} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \times \begin{pmatrix} cos\,\theta & -\,sin\,\theta \\ sin\,\theta & cos\,\theta \end{pmatrix} \tag{6}$$
$$= \begin{pmatrix} a\,cos\,\theta & b\,a\,cos\,\theta - (1/a)\,sin\,\theta \\ a\,sin\,\theta & b\,a\,sin\,\theta + (1/a)\,cos\,\theta \end{pmatrix}$$

  with $det(A_{\sigma_p}) = 1$, $a \in R^*$ and $\theta, b \in R$.
  The Lie algebra of $SL(2, R)$ is denoted by $sl(2, R)$, and is identified with the set of 2×2 matrices and they have a basis provided by e_n:$n = 1, 2, 3$.

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{7}$$

  Lie group theory relies heavily on the Lie algebra of a Lie group since it encodes many of the group's global topological features. Exp, a local diffeomorphism, is also known as exponential mapping.

  In the first step, we do the projection in the space tangent of the A_{σ_p} matrices using the following equation Eq (8) [19].
  Once calculated, the logarithm map of matrices belonging to the lie algebra elements $\ln(A_{\sigma_p}) \in sl(2, R)$ is projected in the tangent space, and we are in the vector space where the matrices must satisfy the following conditions Eq(9):

$$\left\{\ln(A_{\sigma_p}) \in sl(2, R) / Tr(\ln(A_{\sigma_p})) = 0\right\}. \tag{9}$$

  Therefore the exponential mapping of the logarithm mapping $\ln(A_{\sigma_p})$ is expressed as below in Eq(10) [19]:
- A-7: The registration is then performed using the special linear transformation A obtained after the Riemannian calculation and the translation vector B deduced after applying average arithmetic.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad B = \begin{pmatrix} B^x \\ B^y \end{pmatrix} \tag{11}$$

  Finally, we calculate the euclidean distance L_2, which is denoted by:

$$L^2 = \min_{(A,B} = \|Af(l_a) + B - h(l_a)\|^2 \approx e \tag{12}$$

III. Experiments

In this section, we compare the proposed Affine Multi-Scale Curve Registration based on Riemannian metrics to the currently available shape alignment methods and present the recognition rates of each. The MCD dataset is used for testing.

A. MCD image database retrieval

One of the most important uses of the proposed algorithm is in shape registration. Therefore, we will evaluate the Affine Multi-Scale Curve Registration based Riemannian metrics on the Multiview Curve Dataset (MCD) [20], which is made up of 40 shape classes from the MPEG-7 database. Figure

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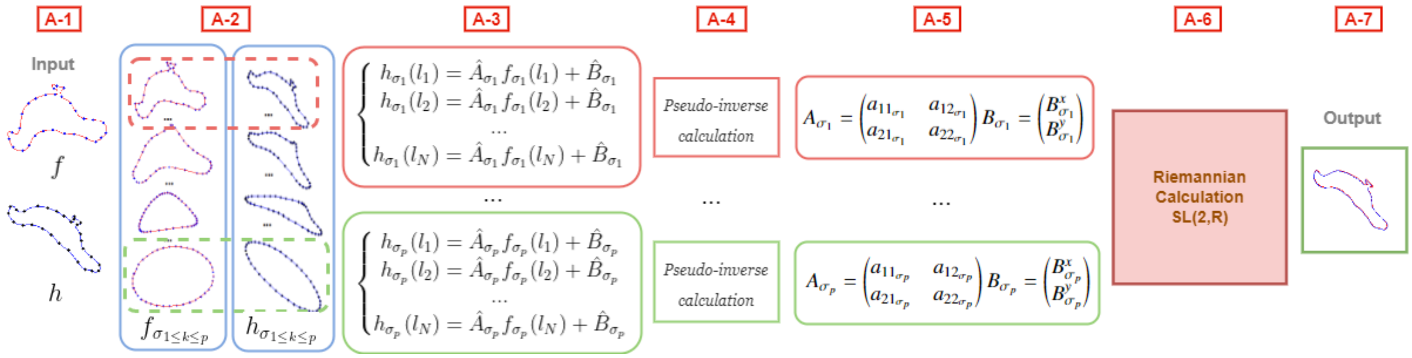
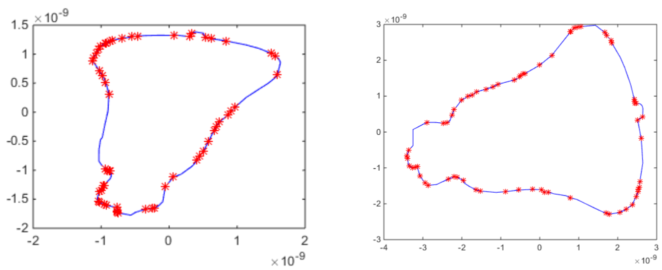


Fig. 1. Workflow of multi-scale contour registration using Riemannian calculation



(a) (b)

Fig. 2. Example of re-sampling shapes with affine arc-length parametrization

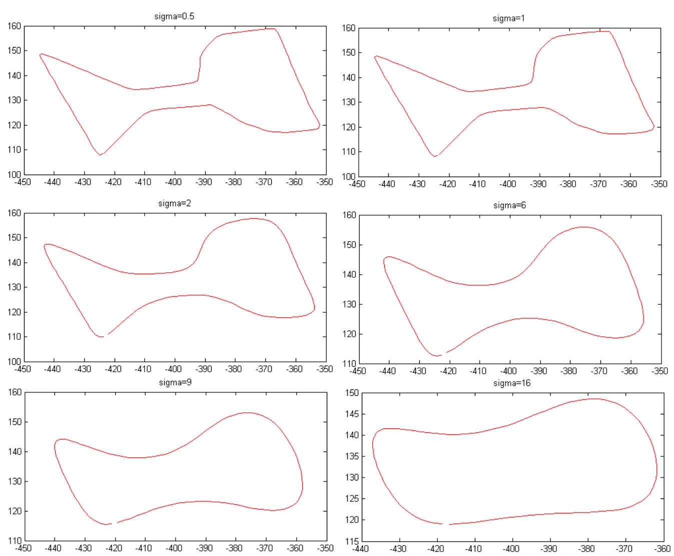


Fig. 3. Example of a convolved shape

4 shows that there are 14 different curves in each of these categories that are distorted in the same way as the original curve.

In Table 1, we compare our methods to some of the current state-of-the-art studies. We discovered that our technique (89.21%) performs better than Arber (41%) [21], SC (56.29%) [22] , Huang (71%) [23], Rube (79%) [24], and Mai (89%) [25]. Our method, on the other hand, is less effective than the methods of fast and non-rigid global registration (92.8%) [26] and ACMA (94%) [16]. Moreover, when compared to AMSCR (96.36%) [15] and AMSCR with Binary-EM (96.58%) [15], our technique demonstrated its limits. The difficulty of the computation in $SL(2, R)$, which will be resolved in future work, demonstrates this limitation clearly.

In Figure 5 we see an example of successfully registered shapes made with our approach.

TABLE I. Retrieval results on the entire MCD dataset

Methods	Average
Arber [21]	41%
SC [22]	56.29%
Huang [23]	71 %
Rube [24]	79 %
Mai [25]	89 %
Our method	**89.21%**
Fast and non-rigid global registration [26]	92.8%
ACMA [16]	94 %
Partial Contour Matching Based on ACSS [27]	95.98%
AMSCR [15]	**96.36 %**
AMSCR with Binary-EM [15]	**96.58 %**

IV. Conclusion and Future Work

In this paper, we suggested a new affine multi-scale curve registration method based on the Riemannian calculation that deals with occlusion and affine transformations. First, the two curves are normalized and smoothed out on different scales. So, for each level, we have several rectangular linear systems. The pseudo-inverse computation is used for each level to compute the special linear transformations A_{σ_p} and translation vectors B_{σ_p}. Afterward, the average of the A_{σ_p} matrices is then calculated using the Riemannian metric in the spatial affine group $SA(2, R)$. After that, the two shapes are lined up, and the euclidean distance L_2 is calculated.

Despite the novelty of the proposed method, the obtained results are not always as good as those of other methods since several numerical challenges remain, such as the choice of the point in the tangent space and the shape's starting point. So, in the future, we will be working to resolve these issues.

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$$if \quad a_{11_{\sigma_p}} + a_{22_{\sigma_p}} \geqslant 2$$

$$\ln(A_{\sigma_p}) = \frac{\ln\left[\left(a_{11_{\sigma_p}} + a_{22_{\sigma_p}} + \sqrt{(a_{11_{\sigma_p}} + a_{22_{\sigma_p}})^2 - 4}\right)/2\right]}{\sqrt{(a_{11_{\sigma_p}} + a_{22_{\sigma_p}})^2 - 4}} \begin{pmatrix} a_{11_{\sigma_p}} - a_{22_{\sigma_p}} & 2a_{12_{\sigma_p}} \\ 2a_{21_{\sigma_p}} & a_{22_{\sigma_p}} - a_{11_{\sigma_p}} \end{pmatrix}$$

$$if -2 < a_{11_{\sigma_p}} + a_{22_{\sigma_p}} \leq 2$$

$$\ln(A_{\sigma_p}) = \frac{\arccos\left[\left(a_{11_{\sigma_p}} + a_{22_{\sigma_p}}\right)/2\right]}{\sqrt{4 - (a_{11_{\sigma_p}} + a_{22_{\sigma_p}})^2}} \begin{pmatrix} a_{11_{\sigma_p}} - a_{22_{\sigma_p}} & 2a_{12_{\sigma_p}} \\ 2a_{21_{\sigma_p}} & a_{22_{\sigma_p}} - a_{11_{\sigma_p}} \end{pmatrix} \tag{8}$$

$$if \quad a^2_{11_{\sigma_p}} + a_{12_{\sigma_p}} a_{21_{\sigma_p}} \geq 0$$

$$A_{\sigma_p} = \cosh\left[\sqrt{a^2_{11_{\sigma_p}} + a_{12_{\sigma_p}} a_{21_{\sigma_p}}}\right] I + \ln\left(A_{\sigma_p}\right) \frac{\sinh\left[\sqrt{a^2_{11_{\sigma_p}} + a_{12_{\sigma_p}} a_{21_{\sigma_p}}}\right]}{\sqrt{a^2_{11_{\sigma_p}} + a_{12_{\sigma_p}} a_{21_{\sigma_p}}}}$$

$$if \quad a^2_{11_{\sigma_p}} + a_{12_{\sigma_p}} a_{21_{\sigma_p}} \leq 0$$

$$A_{\sigma_p} = \cos\left[\sqrt{-a^2_{11_{\sigma_p}} - a_{12_{\sigma_p}} a_{21_{\sigma_p}}}\right] I + \ln\left(A_{\sigma_p}\right) \frac{\sin\left[\sqrt{-a^2_{11_{\sigma_p}} - a_{12_{\sigma_p}} a_{21_{\sigma_p}}}\right]}{\sqrt{-a^2_{11_{\sigma_p}} - a_{12_{\sigma_p}} a_{21_{\sigma_p}}}} \tag{10}$$

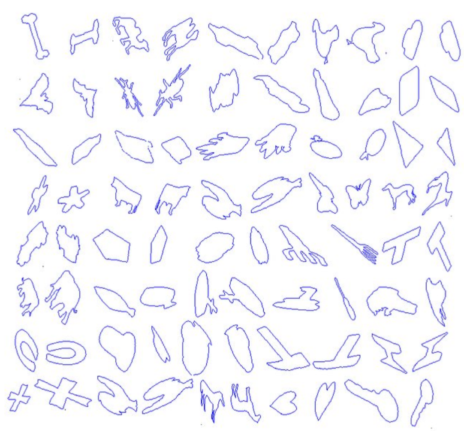

Fig. 4. Different shape images from the MCD dataset, two images from each class.

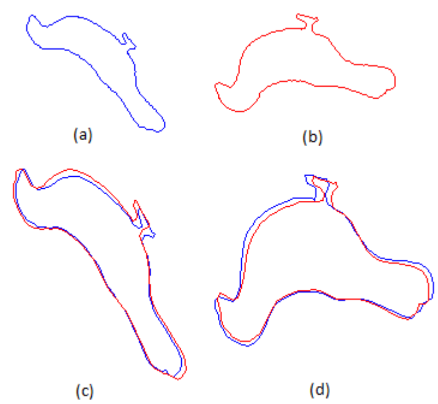

Fig. 5. a and b are the original curves, c, and d display the aligned shapes

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