Fractional Order Variational Approach for Image Denoising and CT Reconstruction

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***Abstract*—Image denoising is a fundamental problem in the area of image processing. The widely applications make it very important to research. Variational method is an efficient way to restore images corrupted by noises. In this paper, we propose a variational model to deal with Gaussian noise and mixed noise. In the proposed model, we use the combination of Total Variation (TV) and Fractional order Total Variation (FTV) as the regularization term. Numerical results show that the proposed model has advantages on recovering image edges and textures. We also generalize our approach to CT image reconstruction by fan beam X-rays from a single radiograph. By a single radiograph, we can reconstruct an axially symmetric object image. The variational model and the algorithm to solve it will be given, and the efficiency of the proposed method will be illustrated by numerical tests.**

Index Terms*—*image denoising; total variation; fractional order; CT; image reconstruction

I. INTRODUCTION

Image denoising is a fundamental problem that has been researched for a long time and yet no clear cut solution exists due to its ill-posedness. To overcome the ill-posedness of the inverse problem arised in image restoration, Total Variation (TV) regularization model was proposed by Rudin, Osher and Fatemmi in 1992 [5]. Since then, variational method was rapidly developed and widely used in image processing tasks. Various variational models were developed for image denoising, deblurring, segmentation, registration and so on. In this work, we focus on image denoising techniques and applications. Total variation regularization model is very efficient on recovering image edges while removing noises. But it often leads to staircase effects on the flat region, and small details, such as textures are often filtered out with noise. Therefore, the improved variational models were considered. Fractional order Total Variation (FTV) regularization has shown its advantages on texture preserving in recent research [1]-[4][6]. In this paper, we propose a new variational model by taking the combination of TV and FTV as regularizer for image noise removal. The proposed variational model is established according to noise type and its characteristics. The noise type deals with Gaussian noise and mixed Poisson and White Spike noise. We also construct the numerical algorithm for proposed model based on Alternating Direction Method of Multipliers (ADMM). Numerical tests show the effectiveness of the proposed method.

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The rest of the short paper is organized as follows: In section 2, we describe the image denoising problem. The mathematical model and algorithm for eliminating Gaussian noise are given. In section 3, we introduce the CT image reconstruction. For axially symmetric objects reconstruction, we illustrate how to establish optimization problem by using the hybrid regularization of TV and FTV for getting efficient reconstruction while suppressing noise. In section 4, we give conclusions and future work.

II. IMAGE DENOISING

Generally, the process of image contaminated by Gaussian noise can be modeled as follows:

$$f = u + n \tag{1}$$

where f is the measured image contaminated by noise n, u is the true image to be recovered. The variational model for image denoising is composed of two parts, the first part is the data fitting term, the second part is the regularization term. The data fitting term depends on the noise type, and the regularization term utilizes the prior knowledge of the unknown image. Therefore, the restoration process results in the model:

$$\min_u \{\frac{1}{2}\|u-f\|_2^2 + \mu\mathcal{R}(u)\}, \tag{2}$$

where μ is the regularization paprameter which balance the data fitting and regularizing. The choice of the regularization term is the key point of the model. For better recovery of the image edges and textures, we propose to use the hybrid regularization of TV and FTV, it takes the following form:

$$\mathcal{R}(u) = \int_\Omega g(x,y)|\nabla u|dxdy + \int_\Omega (1-g(x,y))|\nabla^\alpha u|dxdy \tag{3}$$

Why do we choose the combination of TV and FTV ? As we know, TV regularization is good at edge detection after image restoration, but smears small textures in some region. FTV regularization can protect image textures after restoration. The purpose of taking the combination of TV and FTV as regularizers is to have both advantages of TV and FTV. To balance the weight of TV and FTV, we use a gradient related functional g to judge edges and textures.

We will discuss how to choose fractional order α and the gradient related functional g. To solve the proposed model,

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Fig. 1. Left is a clean image, middle is the noisy one, right is the denoised by our proposed model.

we use Alternating Direction Method of Multipliers. In Figure 1, we give an example of image denoising by our proposed model. The left image is clean, the middle is a noisy one contaminated by Gaussian noise, and the right is the denoised image by our proposed model. We add Gaussian noise to the clean image (left), then we get the noisy image (middle). We use the proposed model (2) to calculate the denoised image denoted by u. In equation (2), f represents the noisy image which is known. The numerical result u is the right image. We can see that the denoised image has got improvement on SNR (signal to noise ratio) and details recovery. For example the hair and the hat edges in Lena image.

III. CT Reconstruction

The image denoising idea can be extended to CT image reconstruction. As we know, CT image reconstruction is very helpful to diagnose the inner features of objects and characteristics of human bodies through x-ay radiographs. In this case, we talk about image reconstruction of axially symmetric objects from a single radiograph by fan beam x-ray. This leads to Abel transform inversion. Suppose the projection matrix is known A, the measured projection data is $\mathbf{d}$, and the object features to be solved is represented by ρ, the CT reconstruction is described by the following mathematical model:

$$\min_{\rho}\{\frac{1}{2}\|A\rho-\mathbf{d}\|_2^2+\mu\mathcal{R}(\rho)\}, \tag{4}$$

$$\mathcal{R}(\rho)=\int_{\Omega}g(r)|\nabla\rho|dr+\int_{\Omega}(1-g(r))|\nabla^{\alpha}\rho|dr \tag{5}$$

Where the function ρ is dependent on variable r which represents the radius of objects with axially symmetry.

CT image reconstruction is an important area for approaching. Especially when the noise type are mixed or even unknown, the reconstruction problem are challenging on mathematical modelling and numerical algorithm designing. If the optimization problem are not convex, there is no guarantee for the global solution of the optimization problem to be reached.

IV. Conclusions

We make an approach on image denoising variational models and algorithms. To deal with the ill-posedness of the inverse problem, we introduce a new regularization term which combines the advantages of TV and FTV. This regularization technique has been applied to CT image reconstruction. For mixed poisson and white spiky noise, we have also explored image restoration models and numerical algorithms. Numerical experiments are given to show the efficiency of our proposed model on recovering image edges and textures while suppressing noises [1]. In the future, we will do further approach according to reviewers' suggestions. At the present stage, our research work is still going on, especially comparisons with other works will be illustrated through numerical tests.

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