Traffic and Monotonic Total-Connected Random Walks of Particles

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***Abstract*–An analytical and simulation models of random walk of particles on a closed one-dimensional lattice are considered. In these models, the particles contained in a cluster move synchronously. The problem is to find the average time interval after which only a cluster remains. This problem is solved with both analytical and simulation approaches. A simulation model is also developed that describes the movement of a particles on a ring with traffic lights. An appropriate analytical model is also developed with some different rules of functioning. The average velocity of the particle is calculated. The results obtained with the simulation and analytical model are compared. Simulations models are also described that are supposed to be developed for the traffic with traffic lights, for multi-lane case, and networks with a periodic structure on that total-connected random walks occur.**

Keywords-stochastic models; random walk; multi-lane traffic.

I. INTRODUCTION

In [1], some analytical and simulations models in terms of random walks are considered that can be interpreted as traffic models. These mathematical models can be interpreted as cellular automata. The models of this class were introduced in [2, 3] and were investigated in a lot of works. The scheme considered in [2, 3] is similar to monotonic random walks on a lattice. The work of Yu. Belyaev and his students [4, 5] is devoted to traffic flows in the underground and contains exact results for one-dimensional random walk (not only monotonic). Appropriate references are given in [1]. Some results in this field have been found in [6–11].

In the present paper, we consider a modification of a model of random walks on a circular lattice. A sequence of adjacent particles is called a cluster. The clusters are separated one from another by empty cells. In the considered model, all the particles within a cluster move synchronously. The number of clusters can decrease and cannot increase. After some time interval with a finite average value, only a cluster remains, if the probability of the cluster movement at a time is less than 1. A problem is solved to find an average time of coming to the state with one cluster. If the lattice is small, then the problem can be solved with an analytical approach. If the number of particles is rather big, then the analytical approach is too complicated. A simulation model is developed that is useful in this case.

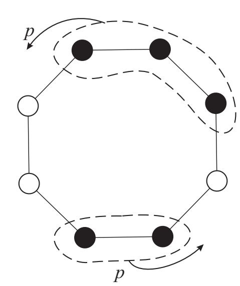

Fig. 1. Total-connected walks on a circle

We call the considered random walks totally-connected, because, in our model, every particle that was contained in a particular cluster always remains contained in the same cluster.

We also develop a simulation model that describes the movement of particles on a ring with traffic lights. An appropriate analytical model is also developed with some different rules of movement. The average velocity of the particle is calculated. The results obtained with the simulation and analytical model are compared.

Simulation models are also described that are supposed to be developed for the traffic with traffic lights, for multi-lane case, and networks with a periodic structure on which total-connected random walks occur. The flow intensity is investigated with these simulation models.

II. THE AVERAGE DURATION OF THE TIME INTERVAL AFTER WHICH ONLY ONE CLUSTER REMAINS

2.1. Consider a closed sequence of cells. The number of cells is equal to n. There are m $(m < n)$ particles. No cell contains more than one particle. The clusters can move one cell forward at discrete times $0, 1, 2, \ldots$ in one direction, Fig. 1. At each discrete time each cluster move one cell forward with the probability p_i, where i is the index of the particle in front of the cluster.

The behavior of the model is described by a Markov chain, [12]. The states of this chain corresponds to the configurations of the particles on the lattice.

Suppose d_i is the average duration of the time interval after which the number of clusters decreases, if at the initial time the chain state is i (the states of the chain are numerated arbitrarily); q_{ij} is the probability that, at the time when the number of clusters decreases, the chain state is j, if the ith state was initial; D_i is the average duration of the time interval after which the number of clusters becomes one. Let A_k be the set of states with k clusters and B_k be the set of states with no more than k clusters.

The problem of calculation of the values d_i and q_{ij} for $i \in A_k$ is reduced to the appropriate problems if it is supposed that the number of cells equals $n-m+k$ and there no cluster containing more than one particle.

We have

$$D_i = \sum_{j \in B_{k-1}} q_{ij} D_j + d_i, \ i \in A_k. \qquad (1)$$

Recurrent formula (1) allows to reduce the problem of calculation of D_i to the problem of calculation of d_i and q_{ij}.

In formula (1), the time interval after which the number of clusters becomes one consists of the time interval after which the number of clusters decreases, and the time interval since the end of the first time interval until the time when the number of clusters becomes one. The average value of the first time interval equals d_i. and the probability that after this interval the chain state is j equals q_{ij}. The average value of the second time interval equals D_j provided it begins itself at the state j. The average value of the total time interval equals D_i. Thus, formula (1) is valid.

In turn, the consideration of this model is reduced to the consideration of a random walk of a particle on a facet of an m-dimensional tetrahedron. Indeed, if x_i is the number of empty cells between the ith particle and the following particle, then the value of the sum $x_1 + \cdots + x_m$ remains constant. The model states correspond to the facet on that this sum remains constant. The model states correspond to the facet $x_1 + \cdots + x_m \leq n - m$ of the tetrahendron $x_1 \geq 0, \ldots, x_m \geq 0$, $x_1 + \cdots + x_m \leq n - m$. The particles coordinates correspond to the lengths of intervals between the particles in the original model, and each coordinate cannot decrease and increase at once more than by one. The problem is to find the average value of the duration of time interval after which the particle comes to the boundary of the facet, where at least one coordinate is equal to zero, and to find the probability that the particle comes to the boundary at a given point. Such problems are solved with an approach described in [12], and these problems are reduced to systems of linear equations. However, the number of the equations can be too big for the system could be solved in practice. Therefore, a simulation model can be useful.

2.2. Suppose $m = 2$. At the initial time there are z empty cells from the particle 2 to the particle 1 in the direction of the particles movement. At each time $0, 1, 2, \ldots$ the particle 1 moves with the probability p_1 and the particle 2 moves with the probability p_2 $(0 < p_1, p_2 < 1)$.

In this case, the problem is reduced to the consideration of random walks on a segment. Denote by d_z the average duration of the interval after which the particles form a cluster. Using an approach described in [12], we get the formulas for d_z.

Proposition 1.

By the above conditions the following formulas are true

$$d_z = \frac{z(n-2-z)}{2p(1-p)}, \ p_1 = p_2 = p,$$

$$d_z = \frac{z}{(1-p_1)p_2 - p_1(1-p_2)} -$$

$$- \frac{(n-2)\left(\left(\frac{(1-p_1)p_2}{p_1(1-p_2)}\right)^z - 1\right)}{((1-p_1)p_2 - p_1(1-p_2))\left(\left(\frac{((1-p_1)p_2}{p_1(1-p_2)}\right)^{n-2} - 1\right)}, \qquad (2)$$

$$p_1 \neq p_2.$$

Denote by d the average duration of the time interval after which the particles are joined.

Proposition 2.

Suppose that all the configurations of two particles on the ring have the same probabilities $p_1 = p_2 = p$. Then

$$\lim_{n \to \infty} \frac{n^2}{12p(1-p)} \cdot \frac{1}{d} = 1,$$

i.e., the asymptotic estimation is true for big values of n

$$d = d(n) =\sim \frac{n^2}{12p(1-p)}.$$

The dependence of $\frac{n^2}{12p(1-p)} \cdot \frac{1}{d}$ on n is showed in Fig.2.

2.3. Suppose now that $m = 3$. TThe case in which the number of particles is more than 3 can be considered similarly. Suppose that the probability of the particle movement at a time is equal to p. Suppose the particle 2 follows the particle 1 in the direction of the movement. Denote by x, y and z the number of cells between the particles 1 and 2, between the particles 2 and 3, and between the particles 3 and 1, appropriately. The model is described by a Markov chain. Each state of this chain corresponds to some point (x, y, z),

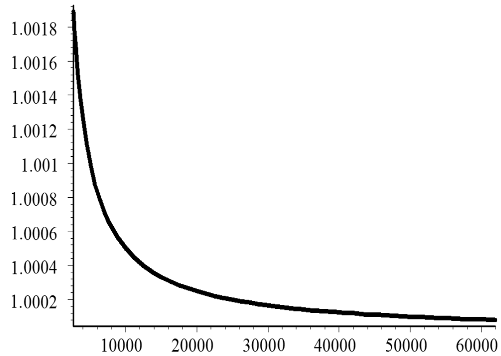

Fig. 2. Asymptotic behavior of the value of d

$x+y+z=n-2$, $x,y,z \geq 0$. Denote by $d(x,y,z)$ the average duration of the time interval after which any two particles form a cluster, if the initial state is (x,y,z), $xyz \neq 0$. Denote by $q(x,y,z;x_0,y_0,z_0)$ the probability that, at the time at that two particles form a cluster, the chain state is (x_0,y_0,z_0), if the initial state is (x,y,z), $xyz \neq 0$, $x_0y_0z_0=0$. The problem is reduced to the investigation of random walks on a facet of a tetrahedron.

Using an approach described in [12], we find *the following system of linear equations, which has a unique solution,*

$$3p(1-p)d(x,y,z)=$$
$$=p^2(1-p)d(x+1,y,z-1)+p^2(1-p)d(x-1,y+1,z)+$$
$$+p^2(1-p)d(x,y-1,z+1)+p(1-p)^2d(x-1,y,z+1)+$$
$$+p(1-p)^2d(x+1,y-1,z)+p(1-p)^2d(x,y+1,z-1)+1,$$
$$xyz \neq 0;$$
$$d(x,y,z)=0,\ \ xyz=0.$$

Each equation of this system corresponds to some set (x,y,z), $x>0$, $y>0$, $z>0$, $x+y+z=n-3$.

Using the same approach, we get *the following system of linear equations, which has also a unique solution,*

$$3p(1-p)q(x,y,z;x_0,y_0,z_0)=$$
$$=p^2(1-p)q(x+1,y,z-1;x_0,y_0,z_0)+$$
$$+p^2(1-p)q(x-1,y+1,z;x_0,y_0,z_0)+$$
$$+p^2(1-p)q(x,y-1,z+1;x_0,y_0,z_0)+$$
$$+p(1-p)^2q(x-1,y,z+1;x_0,y_0,z_0)+$$
$$+p(1-p)^2q(x+1,y-1,z;x_0,y_0,z_0)+$$
$$+p(1-p)^2q(x,y+1,z-1;x_0,y_0,z_0),\ \ xyz \neq 0,$$
$$x_0y_0z_0=0;$$
$$q(x_0,y_0,z_0;x_0,y_0,z_0)=1,\ \ x_0y_0z_0=0;$$
$$q(x,y,z;x_0,y_0,z_0)=0,\ \ (x,y,z) \neq (x_0,y_0,z_0),$$
$$xyz=0,\ \ x_0y_0z_0=0.$$

As above, each equation of this system corresponds to some set (x,y,z), $x>0$, $y>0$, $z>0$, $x+y+z=n-3$.

The proof of the fact that the solution of each of this two systems is unique uses an approach described in [12]. This proof is based on that each inner point (x,y,z) cannot be a point of maximum for $d(x,y,z)$, and, since $d(x,y,z)=0$ for the boundary points $xyz=0$, the the homogeneous system of linear equations has only the zero solution.

III. Simulation model of random walks on a circle

Models that simulate the random walks on a circle have been developed.

There is no cluster that contains more then one particle at the initial state.

The average duration of the interval after which only a cluster remains is investigated.

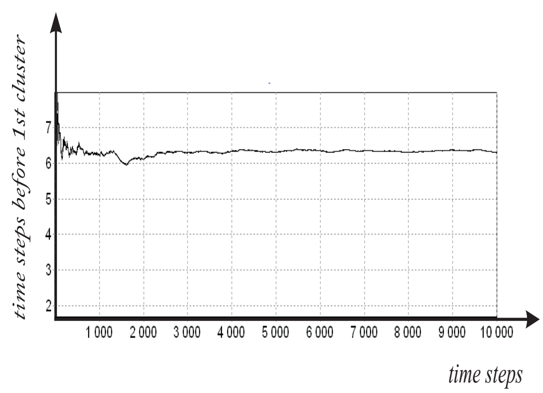

Fig. 3. The average time after which the number of clusters decreases

The dependence of the average number of time steps before the first cluster appears on the number of experiments is represented in Fig. 3. The number of experiments is represented on the x-axis, and the investigated average time is represented on the y-axis. The number of simulation experiments is equal to 10000. We suppose the number of cells equals 20, the number of particles equals 4. The probability that the particle moves at a fixed time is equal to 0.5. The initial distribution of particles is uniform, i.e., before the start of simulation, the particles are inserted into the cells one after another, and the probability that the particle is inserted to a given cell is the same for all the empty cells.

The dependence of the average number of time steps before the first cluster appears on the flow density is represented in Fig. 4. The flow density, i.e., the ratio of the number of particles to the number of cells is represented on the x-axis, and the investigated average time is represented on the y-axis. The number of simulation experiments is equal to 100000 for each density value. We suppose the number of cells equals 100. The probability that the particle moves at a fixed time is equal to 0.5. The initial distribution of particles is uniform as it described above.

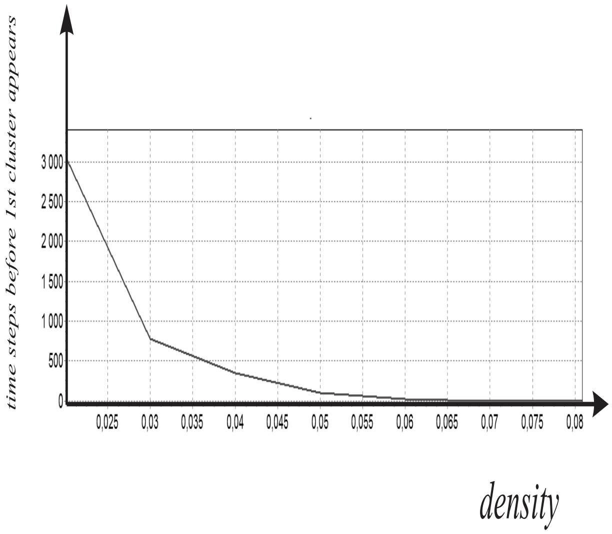

Fig. 4. The average number of time steps before the first new cluster appears

IV. Total-connected walks of a finite number of particles on a straight line

Consider also monotonic total-connected walks of particles on an infinite one-dimensional lattice.

Suppose there is a finite number of particles on a straight line. Suppose there are m clusters. The probability that the ith cluster moves at a time is equal to p_i, $i = 1, \dots, m$. (The first cluster is ahead. The $(i+1)th$ cluster follows the ith cluster, $i = 1, \dots, m-1$.) Then at each time the difference between the coordinates of the first and the last particle increases by one with the probability $p_1(1-p_m)$, decreases by one with the probability $p_m(1-p_1)$, and does not change with the probability $p_1p_m + (1-p_1)(1-p_m)$.

The problem is reduced to the results on the symmetric random walk of a particle on a straight line described in [12, 13].

In [13], a one-dimensional random walk of a particle on a lattice is considered. At each discrete time the particle with probability p moves by one position to the right and with probability q moves by one position to the left, $p + q = 1$. It is proved, [12, 13], that, if $p = q = 1/2$, then the particle returns to a given position with the probability 1 after a finite time interval, but the duration of this interval is infinite. If $p > q$, then with a positive probability the particle shifts to $+\infty$ no returning to the initial position. If $p < q$, then with a positive probability the particle shifts to $+\infty$ no returning to the initial position.

Similarly, we have in our model that, *if $p_1 = p_2 = \cdots = p_m$, then the duration of the interval after which only one cluster remains is finite with the probability 1, but the average duration of this interval is infinite. If $p_1 < p_2 < \cdots < p_m$, then the average duration of the interval after which only one cluster remains is finite. If $p_1 > p_i$ for some i, then this interval is infinite with a positive probability.*

Let us estimate the average duration of the interval after which only one cluster remains, if $p_1 < p_2 < \cdots < p_m$.

Let us consider the case of two clusters with the probabilities of the movement at the current time p_1 and p_2. Let the number of cells between the particles at the initial time be z. The problem is reduced to the consideration of a random walk of a particle that is at the point z in the initial time. The particle moves to the right by one position with the probability p, and the particle moves to the left by one position with the probability q, $p + q = 1$. If $p < q$, then the probability 1 the particle comes with the probability 1 to the position 0 after a finite time interval. The average of the average duration of this interval is equal to $z(q-p)^{-1}$.

In our case, we have to suppose $p = p_i(1-p_{i+1})/(p_{i+1}(1-p_i) + p_i(1-p_{i+1}))$, $q = p_{i+1}(1-p_i)/(p_{i+1}(1-p_i) + p_i(1-p_{i+1}))$. Taking into account that with the probability $1-p_i(1-p_{i+1}) - p_{i+1}(1-p_i)$ the particle remains at the current time at the same point, and therefore the interval for that the particle does not change its position has the average duration

$$\frac{1}{p_{i+1}(1-p_i) + p_i(1-p_{i+1})},$$

we get that in our model, the average duration d_z of the time interval after which two particles form a cluster can be calculated as

$$d_{iz} = \frac{z}{p_{i+1}(1-p_i) - p_i(1-p_{i+1})}. \quad (3)$$

Formula (3) is the limit case of (2) as n tends to ∞.

Suppose now that m is arbitrary, and z_i is the number between the ith and the $(i+1)$th particles, $i = 1, \dots, m-1$. Using (3), we have the following *the upper bound for the average duration of the interval after which only a cluster remains*

$$d < \frac{\sum_{i=1}^{m-1} z_i}{p_2(1-p_1) - p_1(1-p_2)}.$$

We have taken into account that the distance between the particle 1 and the last particle decreases stochastically no less slowly than the distance between particles 1 and 2, because $p_i \geq p_2$, and hence $p_i(1-p_1) - p_1(1-p_i) \geq p_2(1-p_1) - p_1(1-p_2)$, $i = 2, \dots, m-1$.

V. Movement in the presence of an obstacle

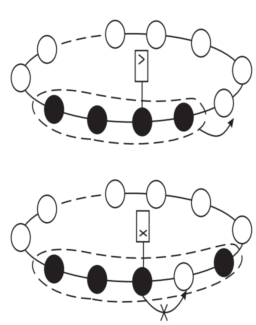

Fig. 5. Movement in the presence of an obstacle

Consider the model that is different from the model considered in Sections 2 and 3, by the fact that an obstacle exists

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on the circle at a given cell (see Fig. 5). This obstacle can be interpreted as the red traffic light. After some time interval the obstacle disappears, and a green phase follows. A cluster is divided if the obstacle appears within the cluster.

The appropriate simulation model is intended to be developed. The main characteristics of movement that have to be investigated in simulation experiments are the average velocity of a particle and the flow intensity. The average velocity v is the average number of the cells that a particle passes per a time unit. The flow intensity q is the average number of particles that passes through a section of the ring per a time unit. For a one-lane model $q = rv$, where r is the flow density, i.e., the ratio of the number of particles to the number of cells.

Consider a model that can describe approximately traffic with traffic lights. Suppose there is a closed sequence of cells. The number of cells is equal to n. There is one particle on the ring. If the particle is in the cell i at the time k, then at the time $k+1$ the particle with the probability a_i, $0 < a_i < 1$, is in the cell $i+1$ and with the probability $1 - a_i$ the particle remains in the cell i, $i = 1, \ldots, n-1$. If the particle is in the cell n at the time k, then at the time $k+1$ the particle with the probability a_n, $0 < a_n < 1$, is in the cell 1 and with the probability $1 - a_n$ the particle remains in the cell n.

Proposition 3.

The formula is true

$$v = \frac{n}{\sum_{j=1}^{n} \frac{1}{a_j}}. \quad (4)$$

Proof. The behavior of the model is described by a Markov chain [12]. Each of the n state of this chain corresponds to the index of the cell that contains the particle. The chain states have stationary probabilities that satisfy the system of equations

$$a_1 p_1 = a_n p_n,$$

$$a_i p_i = a_{i-1} p_{i-1}, \ i = 2, \ldots, n,$$

$$p_1 + \cdots + p_n = 1.$$

This system has a unique solution

$$p_i = \frac{1/a_i}{\sum_{j=1}^{n} \frac{1}{a_j}}, \ i = 1, \ldots, n. \quad (5)$$

The average number of transitions of particles is called the average velocity v of the particle, and

$$v = \sum_{i=1}^{n} p_i a_i. \quad (6)$$

Formula (4) follows from (5) and (6).
Proposition 3 has been proved.

Thus, the average velocity of the particle can be calculated with (4).

Using (4), we can estimate the average velocity of a particle in the model in which there are traffic lights at the cells, and a_i is the ratio of the duration of a green phase to the duration of the total cycle of the traffic lights located at the cell i, $i = 1, \ldots, n$. If there is no traffic lights at the cell i, then $a_i = 1$.

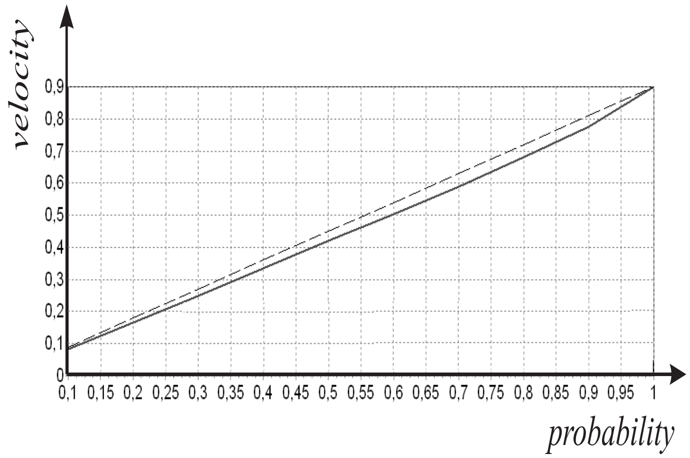

Fig. 6. Comparison of the average velocities found with the simulation and analytical models with some different rules of traffic lights functioning

In Fig. 6, the results of the comparison of the average velocities, found with the simulation and analytical models with some different rules of functioning, is represented. The analytical model is the same as described above. Suppose $n = 9$, $a_1 = p/2$, $a_i = p$, $i = 2, \ldots, n$, where p is a variable, value of which is represented on x-axis. The value of the particle velocity is represented on y-axis. In simulation model the particle moves to the next cell at each time with the probability p. There is a traffic light in the cell 1. The duration of both the green and the red phase is equal to a time unit with the probability 1. The dashed line corresponds to the formula (3) and the solid line corresponds to the simulation experiments. The number of experiments is 100000 for every value of p.

VI. The model of two-lane traffic

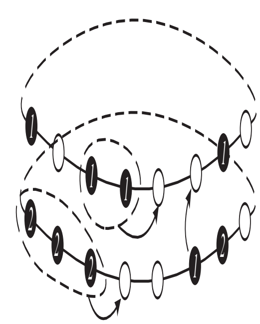

Fig. 7. A model of two-lane movement

Consider a generalization of the model of random walks on a circle.

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Suppose there are two sequences of cells (two lanes) that form a ring of the dimension $2 \times n$. There are m particles on the ring, as shown in Fig. 7.

There are two types of particles. A batch of particles of the same type occupying adjacent cells of the same lane is called a cluster. The clusters of the first type particles are fast, and the clusters of the second type particles are slow. The particles of the same cluster move synchronously. All the particles of a cluster move one cell forward at each discrete time with the probability that depends on the type of the particle in front of the cluster. Clusters can be both united or divided. Two clusters of the same type are united, if they are on the same lane, and one of them catches up the other. Particles contained in a cluster can change the movement lane, and it can occur that the particles of the same cluster become occupying cells on the different lanes. A change of the movement lane can occur if a fast cluster catch up a slow cluster and the appropriate cells are empty, Fig. 8. The particles of the cluster change the movement line one at a time.

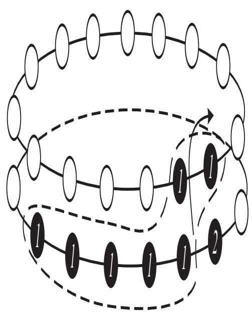

Fig. 8. Particles of a cluster change the movement lane

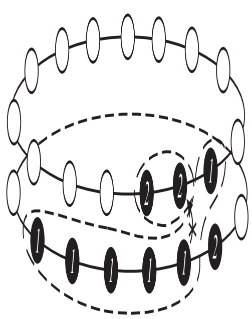

Fig. 9. Situation in that a cluster is divided

A cluster can be divided. Such a situation is showed in Fig. 9.

The main characteristics of the movement that need to be investigated in simulation experiments are the average velocity v of a particle and the flow intensity q: The follow formula is true $q = 2rv$, where q is the flow intensity, v is the average velocity, r is the flow density.

VII. Two rings model

Another generalization of the model of the random walks on a circle is the two rings model.

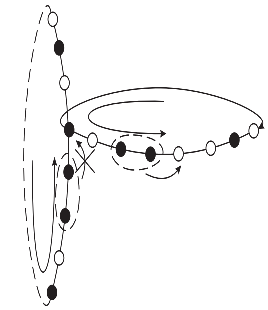

Fig. 10. Two rings model

Suppose there are two sequences of cells (two rings) that have a common cell, Fig. 10. The first ring contains n_1 cells, and the second ring contains n_2 cells. There are m_1 particles on the first ring, and there are m_2 particles on the second ring. The particles contained in the same cluster move synchronously. All particles of the cluster move one cell forward at a discrete time with the probability that can depend on the type of the particle in front of the cluster. Two clusters united if a cluster catch up an other cluster on the same circle.

If two particles can enter into the common cell, then the particle moving on the first ring has the priority. The cluster is divided if a part of the cluster have passed the common cell, and the rest of the cluster is blocked, Fig. 10.

The main characteristics that have to be investigated in simulation experiments are the average velocity of particle and the flow intensity on each ring. The steady probability that a given cell is occupied has to be also estimated.

VIII. Model of a network

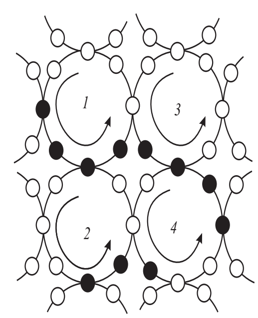

Fig. 11. A fragment of a regular network

Simulation models describing the behavior of networks have to be developed. One of the possible model structures is showed in Fig. 11. The rules of the particles movement are similar to the rules described in the previous sections. If the probability that a cluster moves at each discrete time equals 1 and the flow density is less than 1/2, then all the clusters can move with the velocity equal to 1. If the flow density on each cluster is more than 1/2, the average velocity is less than 1.

IX. Conclusion and Future Work

The analytical and simulation models of random walks of particles have been developed. The particles of the same cluster move synchronously. The results of simulation experiments were represented. The developed models can be used for traffic analysis and optimization [14]. Here are some simulation models that have to be developed:

- Total-connected movement in form of a clustering flow is observed in many cases, e.g., pedestrians, cyclists, traffic flows.

- Clusters are the limit state of the solutions of the system of non-linear ordinary differential equations in the car following model, [15].

- In the methodological sense, cluster objects simplify the investigation of the flow problem on a network.

- Traffic control can increase the number of clusters.

- Problems of clustering on networks are supposed to be investigated.

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