Estimation Quality of High-dimensional Fields in Wireless Sensor Networks

Siyuan Zhou
DET, Politecnico di Torino
Torino, Italy
Email: siyuan.zhou@polito.it

Alessandro Nordio
IEIIT-CNR
Torino, Italy
Email: alessandro.nordio@ieiit.cnr.it

Carla-Fabiana Chiasserini
DET, Politecnico di Torino
Torino, Italy
Email: chiasserini@polito.it

***Abstract*—In the fields of wireless communications, networking and signal processing, systems can be often modeled through a linear relationship involving a random Vandermonde matrix $\mathbf{V}$, and their performance can be characterized through the eigenvalue distribution of the Gram matrix $\mathbf{V}\mathbf{V}^{\mathsf{H}}$. In spite of its key role, little is known about the eigenvalue distribution of such a matrix and only few of its moments are known in closed form. In this work, we obtain a lower and an upper bound to the eigenvalue distribution of $\mathbf{V}\mathbf{V}^{\mathsf{H}}$, as well as an excellent approximation based on entropy maximization. As an application, we consider the case of a wireless sensor network sampling a physical phenomenon to be estimated. We characterize the quality of the estimate through the eigenvalue distribution of $\mathbf{V}\mathbf{V}^{\mathsf{H}}$ by adopting an asymptotic approach, which well suites medium-large scale networks. The proposed method is particularly efficient when dealing with physical phenomena defined over a d-dimensional support, with $d > 2$.**

Keywords*—*sensor networks; Signal estimation; Vandermonde matrices.

I. INTRODUCTION

Recently, random Vandermonde matrices have attracted a great deal of interest since they play an important role in fields such as wireless communications, sensor networks, and image processing. In these contexts, signals estimation systems can often be modeled as [1]–[3]:

$$\mathbf{y} = \mathbf{V}^{\mathsf{H}}\mathbf{a} + \mathbf{n}\,, \tag{1}$$

where the vector $\mathbf{y}$ represents a set of measurements, the vector $\mathbf{a}$ denotes the system input that has to be estimated, $\mathbf{V}$ is a random Vandermonde matrix representing the system transfer function, and $\mathbf{n}$ is the additive noise, uncorrelated with respect to $\mathbf{a}$ and $\mathbf{V}$. Bold lowercase and uppercase letters denote column vectors and matrices, respectively. The conjugate transpose operator is denoted by $(\cdot)^{\mathsf{H}}$, and the identity matrix is denoted by $\mathbf{I}$. In particular, the model in (1) can represent a physical phenomenon sampled by a wireless sensor network composed of nodes randomly deployed on a d-dimensional support. The physical phenomenon is specified by the vector $\mathbf{a}$ which has to be estimated from the set of noisy measurements $\mathbf{y}$, collected by the sensors. In this case, the matrix $\mathbf{V}$ accounts for the positions of the sensor nodes. In [4], it has been shown that the performance of linear estimation techniques can be accurately described through the *eigenvalue distribution* (or *matrix spectrum*) of the Gram matrix associated to $\mathbf{V}$, i.e., $\mathbf{V}\mathbf{V}^{\mathsf{H}}$. In particular, a key role is played by the *asymptotic* spectrum of $\mathbf{V}\mathbf{V}^{\mathsf{H}}$, which is obtained by letting the size of the matrix $\mathbf{V}$ tend to infinity while keeping the ratio of the number of rows to the number of columns constant. Unfortunately, such asymptotic spectrum is still unknown and very few results exist that can shed light on this important issue. For example the results presented in [1], [2] were obtained by using a Monte Carlo approach which, although accurate, turns out to be computationally expensive when the physical phenomenon to be estimated is defined over $d > 2$ dimensions. In this work, we leverage the existing results on the moments of the asymptotic spectrum of $\mathbf{V}\mathbf{V}^{\mathsf{H}}$ and contribute to filling the aforementioned gap by providing (i) a lower and an upper bound to the asymptotic cumulative distribution function of the eigenvalues of $\mathbf{V}\mathbf{V}^{\mathsf{H}}$, and (ii) an approximation of the asymptotic spectrum (both the cumulative and the density functions), which proves to be very accurate. Through such results, we are able to characterize the performance of linear reconstruction techniques by avoiding the need of computationally expensive Monte Carlo methods.

The rest of the paper is organized as follows. First, in Section II we provide some background on the system model and performance metric, as well as on fundamental concepts related to Vandermonde matrices that we use in our analysis. We then present our bounds and approximation of the asymptotic eigenvalue distribution in Section III and Section IV, respectively. Finally, we show numerical results in Section V, and we conclude the paper in Section VI.

II. SIGNAL RECONSTRUCTION IN WSN AND RANDOM VANDERMONDE MATRICES

We consider m sensors sampling a spatially-finite physical phenomenon $s(\mathbf{x})$ (hereinafter also called signal), defined over a d-dimensional hypercube $\mathcal{H} = [0, 1)^d$, $d \geq 1$, and with finite energy. The signal can be approximated by its truncated Fourier series expansion so that the sample of the q-th sensor ($q = 1, \ldots, m$) can be modeled as [1]

$$s_q = s(\mathbf{x}_q) = n^{-d/2} \sum_{\boldsymbol{\ell}} a_{\nu(\boldsymbol{\ell})} \mathrm{e}^{\mathrm{j}2\pi\boldsymbol{\ell}^{\mathsf{T}}\mathbf{x}_q} \tag{2}$$

where n is the approximate bandwidth (per dimension) of the field and $\boldsymbol{\ell} = [\ell_1, \ldots, \ell_d]^{\mathsf{T}}$ is a vector of integers, with $\ell_j = 0, \ldots, n-1$, $j = 1, \ldots, d$. The coefficient $n^{-d/2}$ is a normalization factor and the function $\nu(\boldsymbol{\ell}) = \sum_{j=1}^{d} n^{j-1}\ell_j$, maps uniquely the vector $\boldsymbol{\ell}$ into $\{0, \ldots, n^d - 1\}$. The term $a_{\nu(\boldsymbol{\ell})}$ denotes the $\nu(\boldsymbol{\ell})$-th entry of the vector $\mathbf{a} = [a_0, \ldots, a_{n^d-1}]^{\mathsf{T}}$, which represents the approximated signal spectrum, while the vectors $\mathbf{x}_q \in \mathcal{H}$, represents the coordinate of the q-th sampling point (i.e., the position of the q-th sensor), which is assumed to be known. We assume that $\mathbf{x}_q$, $q = 1, \ldots, m$, are i.i.d. random vectors having a generic continuous distribution over the hypercube $\mathcal{H}$. Also, since in general we do not have any information on the signal spectrum $\mathbf{a}$, we assume it has zero

 ISBN: 978-1-61208-317-9

mean and covariance $\sigma_a^2\mathbf{I}$. Without loss of generality and for normalization reasons, we set $\sigma_a^2 = 1$.

The vector of samples $\mathbf{s} = [s_1, \dots, s_m]^\mathsf{T}$ can be rewritten in a compact form as $\mathbf{s} = \mathbf{V}^\mathsf{H}\mathbf{a}$ where $\mathbf{V}$ is an $n^d \times m$ random Vandermonde matrix whose generic entry

$$(\mathbf{V})_{\nu(\boldsymbol{\ell}),q} = n^{-d/2} \exp\left(-2\pi\mathrm{i}\boldsymbol{\ell}^\mathsf{T}\mathbf{x}_q\right) \tag{3}$$

is randomly distributed on the complex circle of radius $n^{-d/2}$. The vector of noisy samples $\mathbf{y}$ can thus be represented by (1), where $\mathbf{n}$ has zero mean, covariance matrix $\sigma_n^2\mathbf{I}$, and is uncorrelated with $\mathbf{a}$. From the knowledge of $\mathbf{y}$ and of the sampling locations $\mathcal{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_m\}$, the network provides an estimate $\hat{s}(\mathbf{x})$ of the signal $s(\mathbf{x})$. As a performance metric of the signal reconstruction, we take the mean square error (MSE) of the estimate, which can be written in terms of the estimate, $\hat{\mathbf{a}}$, of the signal spectrum $\mathbf{a}$ [1] as

$$\mathcal{M}^{(n,m)} = \mathbb{E}_{\mathbf{a},\mathbf{n},\mathcal{X}}\|\mathbf{a} - \hat{\mathbf{a}}\|^2/n^d \tag{4}$$

where the average is taken with respect to the subscripted random vectors (see [1] for details). In the literature, many estimators for $\mathbf{a}$ have been proposed. Among these, linear estimators such as the zero-forcing (ZF) and the linear minimum MSE (LMMSE) estimators [4] are commonly employed in signal detection and estimation since their analysis can be often carried out analytically. They are given, respectively, as

$$\begin{aligned}\hat{\mathbf{a}}_{\mathrm{ZF}} &= \left(\mathbf{V}\mathbf{V}^\mathsf{H}\right)^{-1}\mathbf{V}\mathbf{y},\\ \hat{\mathbf{a}}_{\mathrm{LMMSE}} &= \left(\mathbf{V}\mathbf{V}^\mathsf{H} + \sigma_n^2\mathbf{I}\right)^{-1}\mathbf{V}\mathbf{y}\,.\end{aligned}$$

By using these estimators, the MSE in (4) becomes

$$\begin{aligned}\mathcal{M}_{\mathrm{ZF}}^{(n,m)} &= \frac{\sigma_n^2}{n^d}\mathsf{Tr}\left\{(\mathbf{V}\mathbf{V}^\mathsf{H})^{-1}\right\}\\ \mathcal{M}_{\mathrm{LMMSE}}^{(n,m)} &= \frac{\sigma_n^2}{n^d}\mathsf{Tr}\left\{(\mathbf{V}\mathbf{V}^\mathsf{H} + \sigma_n^2\mathbf{I})^{-1}\right\}.\end{aligned} \tag{5}$$

When the number of sensors, m, and the number of harmonics, n^d, is large with constant ratio $\beta = n^d/m$, the MSE can be tightly approximated by using the asymptotic MSE defined as

$$\mathcal{M}^\infty = \lim_{n,m\to\infty}\mathcal{M}^{(n,m)}.$$

For the two filters above, the asymptotic MSE is given by

$$\begin{aligned}\mathcal{M}_{\mathrm{ZF}}^\infty &= \sigma_n^2\beta\mathbb{E}_\lambda\left[\lambda^{-1}\right]\\ \mathcal{M}_{\mathrm{LMMSE}}^\infty &= \sigma_n^2\beta\mathbb{E}_\lambda\left[(\lambda + \sigma_n^2\beta)^{-1}\right]\end{aligned} \tag{6}$$

where λ is a random variable distributed as the asymptotic spectrum of $\beta\mathbf{V}\mathbf{V}^\mathsf{H}$ (see [1] for details). The knowledge of the distribution of λ (in the following denoted as $f_\lambda(d,\beta,z)$), for the case where the positions $\mathbf{x}_1, \dots, \mathbf{x}_m$ are i.i.d and uniformly distributed, plays an important role since it allows to compute the asymptotic MSE in (6) also for the case where $\mathbf{x}_1, \dots, \mathbf{x}_m$ are i.i.d. with generic continuous distribution in $\mathcal{H}$ [5]. Due to the scarcity of results on the behavior of $f_\lambda(d,\beta,z)$, whenever needed, such a distribution has to be evaluated through numerical simulations (i.e., by computing the eigenvalues of several realizations of the $n^d \times n^d$ matrix $\beta\mathbf{V}\mathbf{V}^\mathsf{H}$). Clearly, this is feasible only for fairly small values of n, m and becomes impractical already when $d > 2$.

In order to overcome such a problem, we propose to characterize the distribution by exploiting the only results known in closed-form, i.e, its first few moments. Under the assumption of $\mathbf{x}_1, \dots, \mathbf{x}_m$ being uniformly distributed over $\mathcal{H}$, we have that the p-th moment of $f_{u,\lambda}(d,\beta,z)$, defined as

$$\mu_p^{(d)} = \int_0^\infty z^p f_{u,\lambda}(d,\beta,z)\,\mathrm{d}z = \sum_{k=1}^{p}\beta^{p-k}\sum_{\boldsymbol{\omega}\in\Omega_{p,k}} v(\boldsymbol{\omega})^d \tag{7}$$

is a polynomial in the variable β [1]. In (7), $\Omega_{p,k}$ is the set of partitions of the set $\mathcal{P} = \{1, 2, \dots, p\}$ in k subsets, and for any $\boldsymbol{\omega} \in \Omega_{p,k}$, $v(\boldsymbol{\omega}) \in (0,1]$ is a rational number that can be analytically computed from $\boldsymbol{\omega}$ following the procedure described in [1]. Note that the computational complexity of $\mu_p^{(d)}$, increases with the Bell number of p [1] so only few moments are available. For simplicity, in the rest of the paper we omit the superscript (d) in the expression of the moments.

Using the above expressions, in the following sections we derive a lower and an upper bound to the asymptotic cumulative distribution function (CDF)

$$F_\lambda(d,\beta,z) = \int_0^z f_\lambda(d,\beta,t)\,\mathrm{d}t \tag{8}$$

and a tight approximation of $f_\lambda(d,\beta,z)$ and $F_\lambda(d,\beta,z)$.

III. Bounds to $F_\lambda(d,\beta,z)$

Let Λ be a non-negative random variable with CDF $F_\Lambda(z)$. If $\mathbb{E}[\Lambda] < \infty$, then for any $z > 0$ Markov's inequality states that $\mathbb{P}\{\Lambda \geq z\} \leq \mathbb{E}[\Lambda]/z$, i.e., that

$$F_\Lambda(z) = 1 - \mathbb{P}\{\Lambda \geq z\} \geq 1 - \frac{\mathbb{E}[\Lambda]}{z}\,.$$

Now, let $\Lambda = \lambda^p$ where $\lambda \geq 0$ is distributed as the asymptotic spectrum of $\mathbf{V}\mathbf{V}^\mathsf{H}$, and let us define $\zeta = z^p$. It follows that, for any $p > 0$, we have:

$$\begin{aligned}F_\lambda(d,\beta,z) &= 1 - \mathbb{P}\{\lambda \geq z\}\\ &= 1 - \mathbb{P}\{\lambda^p \geq z^p\}\\ &\geq 1 - \frac{\mathbb{E}[\lambda^p]}{z^p}\\ &= 1 - \frac{\mu_p}{z^p}\,.\end{aligned} \tag{9}$$

If the moments μ_p are available for $p = 1, \dots, P$, then by (9) a lower bound to $F_\lambda(d,\beta,z)$ can be obtained as:

$$F_\lambda(d,\beta,z) \geq 1 - \min_p \frac{\mu_p}{z^p} \tag{10}$$

An upper bound to $F_\lambda(d,\beta,z)$ can be derived by using the *left-sided Chebychev inequality* [7, chapter 9.1] stating that for any random variable Λ and $\zeta > 0$

$$\mathbb{P}(\Lambda \leq (1-\zeta)\mathbb{E}[\Lambda]) \leq \frac{\mathbb{E}[\Lambda^2] - \mathbb{E}[\Lambda]^2}{\mathbb{E}[\Lambda^2] + (\zeta^2 - 1)\mathbb{E}[\Lambda]^2}\,. \tag{11}$$

Again, by letting $\Lambda = \lambda^p$ in (11), we obtain

$$\mathbb{P}(\lambda^p \leq (1-\zeta)\mu_p) \leq \frac{\mu_{2p} - \mu_p^2}{\mu_{2p} + (\zeta^2 - 1)\mu_p^2}\,.$$

By defining $z^p = (1-\zeta)\mu_p$ (i.e., $\zeta = 1 - z^p/\mu_p$), we can write:

$$\mathbb{P}(\lambda \leq z) = F_\lambda(d,\beta,z) \leq \frac{\mu_{2p} - \mu_{2p}^2}{\mu_{2p} - 2\mu_p z^p + z^{2p}}\,.$$

 ISBN: 978-1-61208-317-9

The above inequality is valid for $\zeta \geq 0$, i.e., $z \leq \mu_p^{1/p}$. When $z > \mu_p^{1/p}$, we assume $F_\lambda(d,\beta,z) \leq 1$. Then, if the moments μ_p are available for $p = 1,\ldots,\lfloor P/2 \rfloor$, we have:

$$F_\lambda(d,\beta,z) \leq \min_p \begin{cases} \frac{\mu_{2p}-\mu_p^2}{\mu_{2p}-2\mu_p z^p + z^{2p}} & \text{if } z \leq \mu_p^{1/p} \\ 1 & \text{else.} \end{cases} \quad (12)$$

IV. Approximation of $f_\lambda(d,\beta,z)$ and $F_\lambda(d,\beta,z)$

The reconstruction of a probability density function from its moments is known as the *Classical Moment Problem*. Unfortunately, the knowledge of a finite set of moments does not guarantee the uniqueness of the solution [8]. In general a good solution must be selected from a solution space according to some cost metric. A method to solve the problem has been proposed in [9], and it is based on the entropy maximization approach.

In practice, an approximation to $f_\lambda(d,\beta,z)$ can be found by maximizing its entropy, under the constraint that the p-th moment of the distribution must be equal to μ_p, for $p = 1,\ldots,P$. More formally, we have to solve the following constrained optimization problem:

$$\begin{aligned} &\max - \int_0^{+\infty} f_\lambda(d,\beta,z)\log f_\lambda(d,\beta,z)\,\mathrm{d}z \\ &\text{s.t.} \\ &\int_0^{+\infty} z^p f_\lambda(d,\beta,z)\,\mathrm{d}z = \mu_p,\ \ p = 0,\ldots,P\,. \end{aligned} \quad (13)$$

When the number of known moments is low, i.e., for $P \leq 2$, $f_\lambda(d,\beta,z)$ can be reconstructed analytically [11]. If only μ_1 is known, we have $f_\lambda(d,\beta,z) = \exp(-z)$. If μ_1 and μ_2 are known, $f_\lambda(d,\beta,z)$ behaves as Gaussian function and is given by $f_\lambda(d,\beta,z) = \exp(-(1+a+bz+cz^2))$, where a, b and c are solutions of the following multivariate equations:

$$\begin{aligned} &\mathrm{e}^{-1-a}\left(\frac{\frac{b^2}{2}+c}{4}\sqrt{\frac{\pi}{c^5}}\mathrm{e}^{\frac{b^2}{4c}}\left(1-\mathrm{erf}\left(\frac{b}{2\sqrt{c}}\right)\right) - \frac{b}{4c^2}\right) = \mu_2 \\ &\mathrm{e}^{-1-a}\left(\frac{1}{2c} - \frac{b}{4c}\sqrt{\frac{\pi}{c}}\mathrm{e}^{\frac{b^2}{4c}}\left(1-\mathrm{erf}\left(\frac{b}{2\sqrt{c}}\right)\right)\right) = \mu_1 \\ &\mathrm{e}^{-1-a}\sqrt{\frac{\pi}{4c}}\mathrm{e}^{\frac{b^2}{4c}}\left(1-\mathrm{erf}\left(\frac{b}{2\sqrt{c}}\right)\right) = 1 \end{aligned}$$

However, for a larger number of moments, i.e., $P > 2$, (13) is in general intractable analytically. We therefore solve it numerically by resorting to the stable algorithm in [8], which is based on the discretization of the integrals through an N-points Gaussian quadrature rule [10]. The method consists in approximating the function to be integrated with the product of a polynomial function and a weighting function $w(x)$, and then in discretizing the latter two. Specifically, we can write:

$$-\int_0^{+\infty} f_\lambda(d,\beta,z)\log f_\lambda(d,\beta,z)\,\mathrm{d}z \approx -\sum_{j=1}^{N} w_j f_j \log f_j$$

and

$$\int_0^{+\infty} z^p f_\lambda(d,\beta,z)\,\mathrm{d}z \approx \sum_{j=1}^{N} w_j z_j^p f_j,\ \ p = 0,\ldots,P$$

where, for $j = 1,\ldots,N$, $f_j = f_\lambda(d,\beta,z_j)$ while z_j and w_j are, respectively, the abscissae of the Gaussian quadrature rule and the corresponding values of the weighting function.

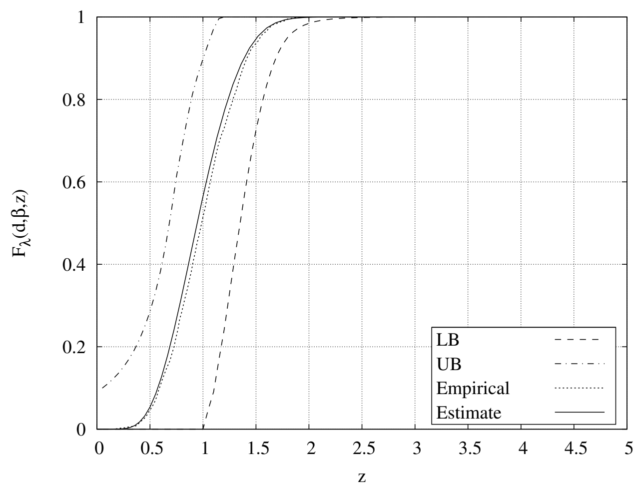

Fig. 1. Comparing bounds, maximum entropy approximation of the CDF, and the empirical distribution, with $d = 1$ and for $\beta = 0.1$.

V. Estimation Performance

We validate our proposed approach by comparing the bounds and approximation against the empirical distribution. The latter has been obtained by computing the eigenvalues of several realizations of $\mathbf{V}\mathbf{V}^{\mathsf{H}}$ for $n = 100$ and a number of sensors $m = n/\beta$, while the maximum entropy approximation has been computed by using the first 12 moments of the asymptotic eigenvalue distribution. All results refer to the case where the phases of the Vandermonde matrix are uniformly distributed over $[0,1)^d$; note, however, that other phase distributions could be considered as well by leveraging the results in [1], [2].

The curves depicted in Figure 1 depict the CDF $F_\lambda(d,\beta,z)$, and have been obtained with $d = 1$ and $\beta = 0.1$. Observe that our bounds follow the behavior of the empirical distribution very well, and there is an excellent match between the latter and the maximum entropy approximation.

Figure 2 compares our approximation to the probability density function, $f_\lambda(d,\beta,z)$, against the empirical results, when $d = 1$ and for $\beta = 0.2$ (top plot) and $\beta = 0.7$ (bottom plot). As expected, in this case the differences between the approximation and the empirical results are more evident than in the case of the CDF, however our approximation still shows to be very tight, even for β as high as 0.7. In addition, the top plot compares the analytical solution of (13), obtained using μ_1 and $\mu_{1,2}$, to our approximation and the empirical results. Clearly, the higher the number of considered moments, the better the approximation accuracy with respect to the empirical results. We also stress that, by definition of β, meaningful values of such a parameter are limited to the $[0,1]$ interval, as the number of sensors (m) should always outnumber the signal harmonics (n^d).

Figure 3 shows the asymptotic MSE, $\mathcal{M}^\infty$, achieved by the ZF and LMMSE filters for $d = 1,4$ and for a noise variance $\sigma_n^2 = 0.01$. The curves were obtained by computing (6) where the distribution of λ was approximated by solving the problem in (13). As expected, the LMMSE filter performs better than the ZF filter since it minimizes the MSE. Also, the figure shows that given number of harmonics per dimension, n,

 ISBN: 978-1-61208-317-9

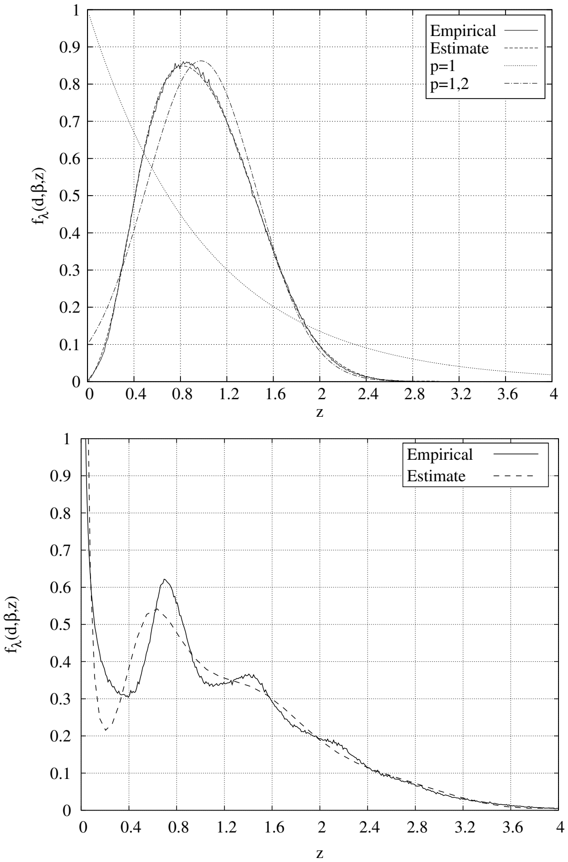

Fig. 2. Comparison between the empirical results and the maximum entropy approximation of the probability density function, with $d = 1$ and for $\beta = 0.2$ (top) and $\beta = 0.7$ (bottom). In the upper plot, it is shown also the analytical solution with p=1, and p=1,2.

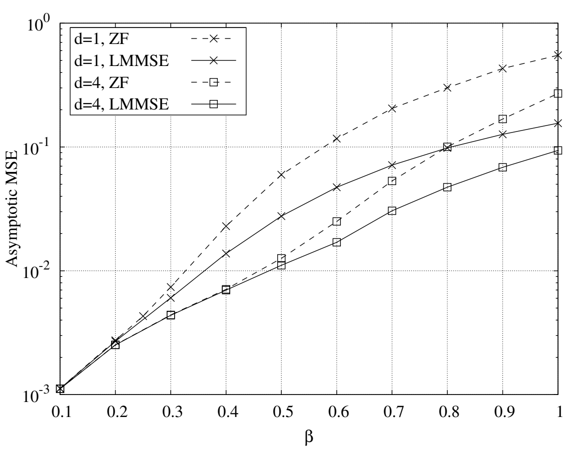

Fig. 3. Comparison between the asymptotic MSE, $\mathcal{M}^\infty$, achieved by the ZF and the LMMSE filters, for $d = 1, 4$ and $\sigma_n^2 = 0.01$.

the asymptotic MSE decreases with the number of available measurements m, i.e., it increases with $\beta = n^d/m$. More importantly, we remark that the maximum entropy approach proposed here, allows to efficiently estimate $f_\lambda(d, \beta, z)$ and $\mathcal{M}^\infty$ for any d, thus avoiding to resort to the numerical computation of the eigenvalues of large matrices.

VI. Conclusions and Future Work

We studied the asymptotic eigenvalue distribution of the Gramian of random Vandermonde matrices, which has an important role in determining the performance of many systems for signal estimation. In particular, we derived a lower and an upper bound to the asymptotic cumulative distribution function. Additionally, we provided an approximation of both the cumulative distribution and the probability density functions, which showed to be very accurate, without applying the cumbersome computation of empirical results.

Our future work will mainly focus on the application of this approximation method to the achievable mutual information of systems, when the channel behavior can be represented by Vandermonde matrix [2]. In this case, we are able to compute the achievable mutual information by deriving the eigenvalue distribution through its moments. Other possible extensions will consider the mutual information in multi-user MIMO systems, and multifold scattering scenarios.

Acknowledgment

This paper was made possible by NPRP grant $\sharp\, 5-782-2-322$ from the Qatar National Research Fund (a member of Qatar Foundation). The statements made herein are solely the responsibility of the authors.

References

[1] A. Nordio, C.-F. Chiasserini, and E. Viterbo, "Reconstruction of multidimensional signals from irregular noisy samples," *IEEE Trans. on Signal Processing,* Vol. 56, No. 9, Sept. 2008, pp. 4274–4285.

[2] O. Ryan, and M. Debbah, "Asymptotic Behavior of Random Vandermonde Matrices with Entries on the Unit Circle", *IEEE Trans. on Information Theory*, Vol. 55, No. 7, July 2009, pp. 3115–3147.

[3] A. Nordio, G. Alfano, C.-F. Chiasserini, and A. M. Tulino, "Asymptotics of Multifold Vandermonde Matrices with Random Entries,", *IEEE Trans. on Signal Processing,* Vol. 59, No. 6, June 2011, pp. 2760–2772.

[4] A. Nordio, C.-F. Chiasserini, and E. Viterbo, "Performance of linear field reconstruction techniques with noise and uncertain sensor locations," *IEEE Trans. on Signal Processing,* Vol. 56, No. 8, Aug. 2008, pp. 3535–3547.

[5] A. Nordio, and C.-F. Chiasserini, "Field Reconstruction in Sensor Networks with Coverage Holes and Packet Losses," *IEEE Trans. on Signal Processing,* Vol. 59, No. 8, Aug. 2011, pp. 3943–3953.

[6] G. H. Tucci, and P. A. Whiting, "Eigenvalue Results for Large Scale Random Vandermonde Matrices with Unit Complex Entries," *IEEE Trans. on Information Theory*, Vol. 57, No. 6, June 2011, pp. 3938–3954.

[7] R. Tempo, G. Calafiore, and F. Dabbene, *Randomized Algorithms for Analysis and Control of Uncertain Systems*, Springer-Verlag, 2005.

[8] K. Bandyopadhyay, K. Bhattacharya, P. Biswas, and D. A. Drabold, "Maximum Entropy and the Problem of Moments: A Stable Algorithm," *Physical Review E 71, 057701*, 2005.

[9] L. R. Mead, and N. Papanicolaou, "Maximum Entropy in the Problem of Moments," *J. Math. Phys. 25, 2404*, Aug. 1984, pp. 2404–2417.

[10] M. Hazewinkel, "Gauss Quadrature Formula," *Encyclopedia of Mathematics*, Springer, 2001.

[11] F. Socheleau, C. Laot, and J. Passerieux, "Concise Derivation of Scattering Function from Channel Entropy Maximization," *IEEE Trans. on Communications*, Vol. 58, No. 11, Nov. 2010, pp. 3098–3103.

 ISBN: 978-1-61208-317-9