# Enhanced Design Conditions for Decentralized State-Space Control of Systems with Relevant Interactions

Dušan Krokavec and Anna Filasová
Department of Cybernetics and Artificial Intelligence
Technical University of Košice, Faculty of Electrical Engineering and Informatics
Košice, Slovakia
dusan.krokavec@tuke.sk, anna.filasova@tuke.sk

***Abstract*—New points of view to the problems concerning the decentralized control of a class of large-scale systems with relevant subsystems interactions are presented in the paper. The problems are transformed into enhanced design conditions through slack matrices until global asymptotic stability of the complete system is pursued using Lyapunov approach. As results, a sufficient condition for the existence is formulated in terms of linear matrix inequalities while the impact of interconnection uncertainties is minimized using $\mathrm{H}_\infty$ approach. The decentralized controllers proved to globally stabilize the system, both in noiseless and noisy conditions.**

***Keywords-Large-scale systems; decentralized control; stabilizing conditions; linear matrix inequalities; $H_\infty$ robust control; control of multi-area power systems.***

## I. INTRODUCTION

Complex large-scale dynamic systems appear in many engineering fields and so, naturally, the control of large-scale systems has been studied by many researchers to provide comprehensive contributions on analytical and computational methods for feedback control design of such systems [2], [17]. Different decentralized control structures were proposed and different algorithms were derived depending on the local state control laws.

If the linear model of a large dynamic system is partitioned into interconnected subsystems, the interactions of the subsystem play significant role in global system stability and, if interactions contain uncertainties, expected performances cannot be attained if the control is designed only for the nominal models. The success of these methods can be improved if the system state are grouped so that subsystem interaction is minimized and the decentralized controllers are optimized with respect to interaction uncertainties. The first usable results for the existence of robust decentralized controllers mostly involve the conditions under which the matrix of interconnections in the considered large-scale system satisfies the prescribed matching condition [20], [23].

Recently, a number of efforts have been made to extend the application of robust control techniques using convex optimization, involving linear matrix inequalities (LMI). It is well known that LMI-based approaches [5] are powerful for a centralized control design, but, in the decentralized case, the control design task may not be oftentimes reducible to a feasibility problem because of existence of control law structural constraints.

To meet modern system requirements, controllers have to quarantine robustness over a wide range of system operating conditions and this further highlights the fact that robustness to interconnections and interaction uncertainties among subsystems is one of the major issues. Applying for power systems control, the most important terms are robustness and a decentralized control structure [15], [24]. The robustness issue arises to deal with uncertainties which mainly come from the varying network topology and the dynamic variation of the load. On the other hand, since a real-time information transfer among subsystems is unfeasible, decentralized controllers have to be exploited. To achieve less-conservative control gains design conditions, norm-bounded unknown uncertainties in subsystem interactions, or nonlinear bounds of interconnections, are included in LMI terms in the design condition formulation [9].

Focusing on the above problems, the paper is sequenced in eight sections and one appendix. Following the introduction in Section I, the second section places the results obtained within the context of existing requests. Section III briefly describes the problems concerning with control of the large-scale dynamical systems with relevant subsystem interactions. The preliminaries, mainly focused on the $H_\infty$ based design approach as well as on the bounded real lemma forms, are presented in Section IV. Section V points out the stability analysis of the controlled system by use of a set of LMIs and Section VI states the newly proposed conditions for the state controller design. Section V illustrates the design task by numerical solutions and system stability analysis and Section VI draws some concluding remarks. Appendix is devoted to a model of the multi-area power systems, used in the illustrative example.

Throughout the paper, the notations is narrowly standard in such way that $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denotes the transpose of the vector $\boldsymbol{x}$ and matrix $\boldsymbol{X}$, respectively, $\boldsymbol{X} = \boldsymbol{X}^T > 0$, $(\geq 0)$, means that $\boldsymbol{X}$ is a symmetric positive definite (semi-definite) matrix, *rank*$(\,\cdot\,)$ remits the rank of a matrix, the symbol $\boldsymbol{I}_n$ indicates the $n$-th order identity matrix, $I\!R$ denotes the set of real numbers, $I\!R^{n\times r}$ refers to the set of all $n\times r$ real matrices, $||\,.\,||$ entails the standard $l_2$-norm and $l_2\langle 0, +\infty)$ connotes the space of random signal over $\langle 0, +\infty)$.

## II. The State of the Art

During the past decades, there has been significant but scattered activity in control of the systems with interactions. A necessary and sufficient condition for solvability, in the case of fixed interconnections, has been found, e.g., in [7], [8], [25], where a homotopic method was used to reduce the control design to a feasibility problem of a bilinear matrix inequality (BMI). Moreover, if the LMI method is adopted by using a single Lyapunov function [3], [19], it leads to very conservative results.

The paper reflects the problems concerning with the system robust stability for one class of disturbed large-scale systems, in the presence of interconnection uncertainties among subsystems. The used approach is concentrated on performance improvement of control systems and is a continuation of the earlier work started in [13], [18], especially motivated by the techniques presented in [4], and improved in [1] with respect to disturbance transfer function norm minimization, the system dynamics improvement and the decentralized control design simplification.

Comparing with the above mentioned articles, the merit of the results proposed in this paper relies on the conservatism reducing through slack matrices incorporation into enhanced design conditions. This represents issues which lead to a newly formulated set of LMIs, giving the sufficient conditions for design of the decentralized controllers, with closed-loop system matrix satisfying the Gershgorin circle theorem [11]. Results are illustrated using the load frequency control model of the multi-area power systems.

## III. Problem Formulation

To formulate the control design task, it is assumed that the subsystems are given adequately to (A.10), (A.11), i.e., it is considered for $i = 1, 2, ..., p$ that

$$\dot{\boldsymbol{q}}_i(t) = \boldsymbol{A}_i\boldsymbol{q}_i(t) + \boldsymbol{b}_i u_i(t) + \sum_{l=1}^{p}\boldsymbol{G}_{il}\boldsymbol{q}_l(t) + \boldsymbol{f}_i d_i(t) \tag{1}$$

$$y_i(t) = \boldsymbol{c}_i^T\boldsymbol{q}_i(t) \tag{2}$$

where $\boldsymbol{q}_i(t) \in I\!R^{n_i}$ is the vector of the state variables of the $i$-th subsystem, $u_i(t), y_i(t) \in I\!R$ are input and output variables of the $i$-th subsystem, respectively, $\boldsymbol{A}_i, \boldsymbol{G}_{il} \in I\!R^{n_i \times n_i}$ are real matrices, $\boldsymbol{b}_i, \boldsymbol{c}_i, \boldsymbol{f}_i \in I\!R^{n_i}$ are real column vectors. The disturbance $d_i(t)$ is a non-anticipative precess, where $\{d(t) \in l_2(\langle 0, \infty); I\!R)$.

It is supposed that all states variables of a subsystem are measured or observed, all subsystems matrix of dynamics $\boldsymbol{A}_i$, $i = 1, 2, ..., p$ are of full rank, all pairs $(\boldsymbol{A}_i, \boldsymbol{b}_i)$ are controllable, and the $i$-th subsystem is controlled by the local state feedback control law

$$u_i(t) = \boldsymbol{k}_i^T\boldsymbol{q}_i(t) \tag{3}$$

where $\boldsymbol{k}_i \in I\!R^{n_i}$ is a constant gain vector.

It is supposed that the interconnections with the uncertainty terms in (1) can be, in general, written as

$$\boldsymbol{G}_i\boldsymbol{h}_i(\boldsymbol{q}(t)) = \sum_{l=1}^{p}\boldsymbol{G}_{il}\boldsymbol{q}_l(t) \tag{4}$$

where $\boldsymbol{h}_i(\boldsymbol{q}(t)) \in I\!R^{n_i}$ is a vector function, satisfying the inequality

$$\boldsymbol{h}_i^T(\boldsymbol{q}(t))\boldsymbol{h}_i(\boldsymbol{q}(t)) \le \varepsilon_i^{-1}\boldsymbol{q}^T(t)\boldsymbol{w}_i^T\boldsymbol{w}_i\boldsymbol{q}(t) \tag{5}$$

where $\varepsilon_i^{-1} > 0$, $\varepsilon_i \in I\!R$ is a scalar parameter, related to interconnection uncertainties, and $\boldsymbol{w}_i$ are constant vectors of appropriate dimensions.

Using the overall system state variable vector $\boldsymbol{q}(t)$, defined as follows

$$\boldsymbol{q}^T(t) = \begin{bmatrix} \boldsymbol{q}_1^T(t) & \boldsymbol{q}_2^T(t) & \cdots & \boldsymbol{q}_p^T(t) \end{bmatrix} \tag{6}$$

then (5) can be rewritten as

$$\begin{aligned} \sum_{l=1}^{p}\boldsymbol{h}_l^T(\boldsymbol{q}(t))\boldsymbol{h}_l(\boldsymbol{q}(t)) = \boldsymbol{h}^T(\boldsymbol{q}(t))\boldsymbol{h}(\boldsymbol{q}(t)) \le \\ \le \boldsymbol{q}^T(t)\left[\sum_{l=1}^{p}\varepsilon_l^{-1}\boldsymbol{w}_l^T\boldsymbol{w}_l\right]\boldsymbol{q}(t) \end{aligned} \tag{7}$$

The global system model with the subsystem interactions takes now the form

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{G}\boldsymbol{h}(\boldsymbol{q}(t)) + \boldsymbol{F}\boldsymbol{d}(t) \tag{8}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) \tag{9}$$

where

$$\boldsymbol{y}^T(t) = [\ y_1(t) \quad y_2(t) \quad \cdots \quad y_p(t)\ ] \tag{10}$$

$$\boldsymbol{u}^T(t) = [\ u_1(t) \quad u_2(t) \quad \cdots \quad u_p(t)\ ] \tag{11}$$

$$\boldsymbol{d}^T(t) = [\ d_1(t) \quad d_2(t) \quad \cdots \quad \boldsymbol{d}_p(t)\ ] \tag{12}$$

$$\boldsymbol{A} = \mathrm{diag}\begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 & \cdots & \boldsymbol{A}_p \end{bmatrix} \tag{13}$$

$$\boldsymbol{B} = \mathrm{diag}\begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_p \end{bmatrix} \tag{14}$$

$$\boldsymbol{G} = \mathrm{diag}\begin{bmatrix} \boldsymbol{G}_1 & \boldsymbol{G}_2 & \cdots & \boldsymbol{G}_p \end{bmatrix} \tag{15}$$

$$\boldsymbol{F} = \mathrm{diag}\begin{bmatrix} \boldsymbol{f}_1 & \boldsymbol{f}_1 & \cdots & \boldsymbol{f}_p \end{bmatrix} \tag{16}$$

$$\boldsymbol{C} = \mathrm{diag}\begin{bmatrix} \boldsymbol{c}_1^T & \boldsymbol{c}_1^T & \cdots & \boldsymbol{c}_p^T \end{bmatrix} \tag{17}$$

where $\boldsymbol{q}(t) \in I\!R^n$, $\boldsymbol{u}(t), \boldsymbol{y}(t) \in I\!R^r$, $\boldsymbol{A}, \boldsymbol{G} \in I\!R^{n \times n}$, $\boldsymbol{B}, \boldsymbol{F} \in I\!R^{n \times r}$, $\boldsymbol{C} \in I\!R^{r \times n}$ and $\sum_{i=1}^{p} n_i = n$.

The goal is the parameter design of the control law for overall system

$$\boldsymbol{u}(t) = \boldsymbol{K}\boldsymbol{q}(t) \tag{18}$$

where $\boldsymbol{K} \in I\!R^{r \times n}$,

$$\boldsymbol{K} = \mathrm{diag}\begin{bmatrix} \boldsymbol{k}_1^T & \boldsymbol{k}_2^T & \cdots & \boldsymbol{k}_p^T \end{bmatrix} \tag{19}$$

in such way that the controlled global large-scale system is stable.

## IV. PRELIMINARY RESULTS

The main purpose of this section is to present the concept of system quadratic performance, based on the H$_\infty$ norm of e system transfer matrix. In that sense are proven and exploited the following results.

*Proposition 1:* If $\boldsymbol{M}$, $\boldsymbol{N}$ are matrices of appropriate dimensions, and $\boldsymbol{X}$ is a symmetric positive definite matrix of proper dimension, then

$$\boldsymbol{M}^T\boldsymbol{N}+\boldsymbol{N}^T\boldsymbol{M}\leq\boldsymbol{N}^T\boldsymbol{X}\boldsymbol{N}+\boldsymbol{M}^T\boldsymbol{X}^{-1}\boldsymbol{M} \tag{20}$$

*Proof:* [12] Since $\boldsymbol{X}=\boldsymbol{X}^T>0$, then

$$\big(\boldsymbol{X}^{-\frac{1}{2}}\boldsymbol{M}-\boldsymbol{X}^{\frac{1}{2}}\boldsymbol{N}\big)^T\big(\boldsymbol{X}^{-\frac{1}{2}}\boldsymbol{M}-\boldsymbol{X}^{\frac{1}{2}}\boldsymbol{N}\big)\geq 0 \tag{21}$$

$$\boldsymbol{M}^T\boldsymbol{X}^{-1}\boldsymbol{M}+\boldsymbol{N}^T\boldsymbol{X}\boldsymbol{N}-\boldsymbol{M}^T\boldsymbol{N}-\boldsymbol{N}^T\boldsymbol{M}\geq 0 \tag{22}$$

It is evident that (22) implies (20). This concludes the proof. ■

*Definition 1:* Let a linear multi input and multi output (MIMO) system is described in the the state-space form by the equation

$$\dot{\boldsymbol{q}}(t)=\boldsymbol{A}\boldsymbol{q}(t)+\boldsymbol{B}\boldsymbol{u}(t) \tag{23}$$

and the output relation

$$\boldsymbol{y}(t)=\boldsymbol{C}\boldsymbol{q}(t)+\boldsymbol{D}\boldsymbol{u}(t) \tag{24}$$

where $\boldsymbol{q}(t)\in I\!R^n$, $\boldsymbol{u}(t)\in I\!R^r$, and $\boldsymbol{y}(t)\in I\!R^m$ are vectors of the state, input and output variables, respectively, and $\boldsymbol{A}\in I\!R^{n\times n}$, $\boldsymbol{B}\in I\!R^{n\times r}$, $\boldsymbol{C}\in I\!R^{m\times n}$ and $\boldsymbol{D}\in I\!R^{m\times r}$ are real matrices. Then the transfer function matrix $\boldsymbol{G}(s)$ of the system (23), (24) is

$$\boldsymbol{G}(s)=\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D} \tag{25}$$

Note, this definition is used only in this section.

The proof of announced lemmas in this section is based on the following result (see, e.g., the proof of Theorem 1 in [12]).

*Proposition 2:* (quadratic performance) If a stable system is described by the transfer function matrix (25) of the dimension $m\times r$, there exists such positive $\gamma\in I\!R$ that

$$\int_0^\infty(\boldsymbol{y}^T(v)\boldsymbol{y}(v)-\gamma\boldsymbol{u}^T(v)\boldsymbol{u}(v))\mathrm{d}v>0 \tag{26}$$

where $\boldsymbol{y}(t)\in I\!R^m$ is the vector of the system output variables, $\boldsymbol{u}(t)\in I\!R^r$ is the vector of the system input variables and $\gamma$ is an upper bound of square of the H$_\infty$ norm of (25).

*Proof:* It is evident, from (25), that

$$\widetilde{\boldsymbol{y}}(s)=\boldsymbol{G}(s)\widetilde{\boldsymbol{u}}(s) \tag{27}$$

where $\widetilde{\boldsymbol{y}}(s)$, $\widetilde{\boldsymbol{u}}(s)$ stands for the Laplace transform of $m$ dimensional output vector and $r$ dimensional input vector, respectively. Then (27) implies

$$\|\widetilde{\boldsymbol{y}}(s)\|\leq\|\boldsymbol{G}(s)\|\|\widetilde{\boldsymbol{u}}(s)\| \tag{28}$$

with $\|\boldsymbol{G}(s)\|$ standing for the $H_2$ norm of the system transfer function matrix $\boldsymbol{G}(s)$. Since $H_\infty$ norm property states

$$\frac{1}{\sqrt{m}}\|\boldsymbol{G}(s)\|_\infty\leq\|\boldsymbol{G}(s)\|\leq\sqrt{r}\|\boldsymbol{G}(s)\|_\infty \tag{29}$$

where $\|\boldsymbol{G}(s)\|_\infty$ is the $H_\infty$ norm of the system transfer function matrix $\boldsymbol{G}(s)$, using the notation $\|\boldsymbol{G}(s)\|_\infty=\sqrt{\gamma}$, the inequality (29) can be rewritten as

$$0<\frac{1}{\sqrt{m}}\leq\frac{\|\widetilde{\boldsymbol{y}}(s)\|}{\sqrt{\gamma}\|\widetilde{\boldsymbol{u}}(s)\|}\leq\frac{\|\boldsymbol{G}(s)\|}{\sqrt{\gamma}}\leq\sqrt{r} \tag{30}$$

Thus, based on Parceval's theorem, (30) gives for $m\geq 1$

$$1\leq\frac{\|\widetilde{\boldsymbol{y}}(s)\|}{\sqrt{\gamma}\|\widetilde{\boldsymbol{u}}(s)\|}=\frac{\Big(\int\limits_0^\infty\boldsymbol{y}^T(v)\boldsymbol{y}(v)\mathrm{d}v\Big)^{\frac{1}{2}}}{\sqrt{\gamma}\Big(\int\limits_0^\infty\boldsymbol{u}^T(v)\boldsymbol{u}(v)\mathrm{d}v\Big)^{\frac{1}{2}}} \tag{31}$$

and, subsequently, it yields

$$\int_0^\infty\boldsymbol{y}^T(v)\boldsymbol{y}(v)\mathrm{d}v-\gamma\int_0^\infty\boldsymbol{u}^T(v)\boldsymbol{u}(v)\mathrm{d}v\geq 0 \tag{32}$$

that is the mapping from $\boldsymbol{u}(t)$ to $\boldsymbol{y}(t)$ is said to have the $H_\infty$ norm less than $\sqrt{\gamma}$.

It is evident that (32) implies (26). This concludes the proof. ■

Before exploiting the principle of quadratic performance in the design of control (18), the following bounded real lemmas for the system (23), (24) are recalled.

*Lemma 1:* (bounded real lemma) System described by (23), (24) is asymptotically stable with the quadratic performance $\sqrt{\gamma}$ if there exist a symmetric positive definite matrix $\boldsymbol{P}\in I\!R^{n\times n}$ and a positive scalar $\gamma\in I\!R$ such that

$$\boldsymbol{P}=\boldsymbol{P}^T>0,\qquad\gamma>0 \tag{33}$$

$$\begin{bmatrix}\boldsymbol{P}\boldsymbol{A}+\boldsymbol{A}^T\boldsymbol{P} & \boldsymbol{P}\boldsymbol{B} & \boldsymbol{C}^T\\ * & -\gamma\boldsymbol{I}_r & \boldsymbol{D}^T\\ * & * & -\boldsymbol{I}_m\end{bmatrix}<0 \tag{34}$$

where $\boldsymbol{I}_r\in I\!R^{r\times r}$, $\boldsymbol{I}_m\in I\!R^{m\times m}$ are identity matrices of given dimensions, respectively.

Here, and hereafter, $*$ denotes the symmetric item in a symmetric matrix.

*Proof:* (compare [5], [18]) Since overall system (23), (24) is linear in $\boldsymbol{q}(t)$, using the Krasovskii theorem (see, e.g., [16]) and considering (32), the Lyapunov function $v(\boldsymbol{q}(t))$ can be considered as

$$\begin{aligned}v(\boldsymbol{q}(t))&=\boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{q}(t)+\\&+\int\limits_0^t(\boldsymbol{y}^T(v)\boldsymbol{y}(v)-\gamma\boldsymbol{r}^T(v)\boldsymbol{u}(v))\mathrm{d}v>0\end{aligned} \tag{35}$$

where $\boldsymbol{P}=\boldsymbol{P}^T>0$, $\gamma>0$.

Thus, evaluating the derivative of $v(\boldsymbol{q}(t))$ with respect to $t$ along a system trajectory, it yields

$$\begin{aligned}\dot{v}(\boldsymbol{q}(t))&=\dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t)+\boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t)+\\&+\boldsymbol{y}^T(t)\boldsymbol{y}(t)-\gamma\boldsymbol{u}^T(t)\boldsymbol{u}(t)<0\end{aligned} \tag{36}$$

Therefore, the substitution of (23), (24) into (36) gives

$$\begin{aligned}\dot{v}(\boldsymbol{q}(t))&=(\boldsymbol{A}\boldsymbol{q}(t)+\boldsymbol{B}\boldsymbol{u}(t))^T\boldsymbol{P}\boldsymbol{q}(t)+\\&+\boldsymbol{q}^T(t)\boldsymbol{P}(\boldsymbol{A}\boldsymbol{q}(t)+\boldsymbol{B}\boldsymbol{u}(t))-\gamma\boldsymbol{u}^T(t)\boldsymbol{u}(t)+\\&+(\boldsymbol{C}\boldsymbol{q}(t)+\boldsymbol{D}\boldsymbol{u}(t))^T(\boldsymbol{C}\boldsymbol{q}(t)+\boldsymbol{D}\boldsymbol{u}(t))<0\end{aligned} \tag{37}$$

and with the notation

$$\boldsymbol{q}_c^T(t) = \left[\begin{array}{cc} \boldsymbol{q}^T(t) & \boldsymbol{u}^T(t) \end{array}\right] \tag{38}$$

it is obtained

$$\dot{v}(\boldsymbol{q}(t)) = \boldsymbol{q}_c^T(t)\boldsymbol{P}_c\,\boldsymbol{q}_c(t) < 0 \tag{39}$$

where

$$\boldsymbol{P}_c = \left[\begin{array}{cc} \boldsymbol{A}^T\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A} & \boldsymbol{P}\boldsymbol{B} \\ * & -\gamma\boldsymbol{I}_r \end{array}\right] + \left[\begin{array}{cc} \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{C}^T\boldsymbol{D} \\ * & \boldsymbol{D}^T\boldsymbol{D} \end{array}\right] < 0 \tag{40}$$

Since

$$\left[\begin{array}{cc} \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{C}^T\boldsymbol{D} \\ * & \boldsymbol{D}^T\boldsymbol{D} \end{array}\right] = \left[\begin{array}{c} \boldsymbol{C}^T \\ \boldsymbol{D}^T \end{array}\right]\left[\begin{array}{cc} \boldsymbol{C} & \boldsymbol{D} \end{array}\right] \geq 0 \tag{41}$$

applying Schur complement property to (41), then (40) implies (34). This concludes the proof. ■

From this results, the stability problem is reduced to find a Lyapunov matrix $\boldsymbol{P}$ and a parameter $\gamma$ to stabilize the system and to guarantee the $H_\infty$ norm attenuation between $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$.

*Lemma 2:* (enhanced bounded real lemma) System described by (23), (24) is asymptotically stable with the quadratic performance $\sqrt{\gamma}$ if for given positive $\delta \in I\!R$ there exist symmetric positive definite matrices $\boldsymbol{P}, \boldsymbol{S} \in I\!R^{n\times n}$, and a positive scalar $\gamma \in I\!R$ such that

$$\boldsymbol{P} = \boldsymbol{P}^T > 0, \quad \boldsymbol{S} = \boldsymbol{S}^T > 0, \quad \gamma > 0 \tag{42}$$

$$\left[\begin{array}{cccc} \boldsymbol{S}_1\boldsymbol{A}+\boldsymbol{A}^T\boldsymbol{S} & \boldsymbol{S}\boldsymbol{B} & \boldsymbol{P}-\boldsymbol{S}+\delta\boldsymbol{A}^T\boldsymbol{S} & \boldsymbol{C}^T \\ * & -\gamma\boldsymbol{I}_r & \delta\boldsymbol{B}^T\boldsymbol{S} & \boldsymbol{D}^T \\ * & * & -2\delta\boldsymbol{S} & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{array}\right] < 0 \tag{43}$$

*Proof:* (compare, e.g., [13]) Since (23) implies

$$\boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) - \dot{\boldsymbol{q}}(t) = \boldsymbol{0} \tag{44}$$

then with positive definite symmetric matrices $\boldsymbol{S}_1, \boldsymbol{S}_2 \in I\!R^{n\times n}$ it yields

$$\left(\boldsymbol{q}^T(t)\boldsymbol{S}_1 + \dot{\boldsymbol{q}}^T(t)\boldsymbol{S}_2\right)\left(\boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) - \dot{\boldsymbol{q}}(t)\right) = 0 \tag{45}$$

Thus, adding (45) as well as the transpose of (45) to (36) and substituting (24) in (36) results in

$$\begin{aligned} \dot{v}(\boldsymbol{q}(t)) = {} & \dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t) - \gamma\boldsymbol{u}^T(t)\boldsymbol{u}(t) + \\ & +(\boldsymbol{C}\boldsymbol{q}(t)+\boldsymbol{D}\boldsymbol{u}(t))^T(\boldsymbol{C}\boldsymbol{q}(t)+\boldsymbol{D}\boldsymbol{u}(t)) + \\ & +(\boldsymbol{A}\boldsymbol{q}(t)+\boldsymbol{B}\boldsymbol{u}(t)-\dot{\boldsymbol{q}}(t))^T(\boldsymbol{S}_1\boldsymbol{q}(t)+\boldsymbol{S}_2\dot{\boldsymbol{q}}(t)) + \\ & +(\boldsymbol{q}^T(t)\boldsymbol{S}_1+\dot{\boldsymbol{q}}^T(t)\boldsymbol{S}_2)(\boldsymbol{A}\boldsymbol{q}(t)-\boldsymbol{B}\boldsymbol{u}(t)-\dot{\boldsymbol{q}}(t)) < 0 \end{aligned} \tag{46}$$

Using the notation

$$\boldsymbol{q}_c^{\circ T}(t) = \left[\begin{array}{ccc} \boldsymbol{q}^T(t) & \boldsymbol{u}^T(t) & \dot{\boldsymbol{q}}^T(t) \end{array}\right] \tag{47}$$

the inequality (46) can be written as

$$\dot{v}(\boldsymbol{q}(t)) = \boldsymbol{q}_c^{\circ T}(t)\boldsymbol{P}_c^\circ\boldsymbol{q}_c^\circ(t) < 0 \tag{48}$$

where

$$\boldsymbol{P}_c^\circ = \boldsymbol{P}_{c1}^\circ + \boldsymbol{P}_{c2}^\circ < 0 \tag{49}$$

$$\boldsymbol{P}_{c1}^\circ = \left[\begin{array}{ccc} \boldsymbol{S}_1\boldsymbol{A}+\boldsymbol{A}^T\boldsymbol{S}_1^T & \boldsymbol{S}_1\boldsymbol{B} & \boldsymbol{P}-\boldsymbol{S}_1+\boldsymbol{A}^T\boldsymbol{S}_2 \\ * & -\gamma\boldsymbol{I}_r & \boldsymbol{B}^T\boldsymbol{S}_2 \\ * & * & -2\boldsymbol{S}_2 \end{array}\right] < 0 \tag{50}$$

$$\boldsymbol{P}_{c2}^\circ = \left[\begin{array}{ccc} \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{C}^T\boldsymbol{D} & \boldsymbol{0} \\ * & \boldsymbol{D}^T\boldsymbol{D} & \boldsymbol{0} \\ * & * & \boldsymbol{0} \end{array}\right] \tag{51}$$

Thus, setting

$$\boldsymbol{S}_1 = \boldsymbol{S}, \qquad \boldsymbol{S}_2 = \delta\boldsymbol{S} \tag{52}$$

and applying analogously Schur complement property to (51), then (49) implies (43). This concludes the proof. ■

The consequence of this Lemma is that of separating the Lyapunov matrix $\boldsymbol{P}$ from the system matric parameters, i.e., there is no product $\boldsymbol{PA}$, $\boldsymbol{PB}$ in the LMIs, which substantially reduces conservatism of solutions, especially if the system is linear with polytopic uncertainties.

Conversely, in the Lemma, a positive real scalar $\delta$ is involved in the LMIs as a prescribed constant design parameter. The procedure of adding scalar in LMIs has been widely explored in literature (see, e.g., [22]). Moreover, such a parameterization is often needed when converting BMI into linear ones.

## V. State Control Design

Algorithms for solutions to (18), which includes the design in the sense of this paper, are the subject of this section.

*Proposition 3:* [1] The autonomous system from (8) is asymptotically stable with bounded quadratic performance if there exist symmetric positive definite matrices $\boldsymbol{P}_i \in I\!R^{n_i\times n_i}$ and positive scalars $\gamma_i, \lambda_i, \varepsilon_i \in I\!R$ such that for $i = 1, 2, \ldots, p$

$$\boldsymbol{P}_i = \boldsymbol{P}_i > 0,\ \gamma_i > 0,\ \lambda_i > 0,\ \varepsilon_i > 0 \tag{53}$$

$$\left[\begin{array}{cccccccc} \boldsymbol{\Phi} & \boldsymbol{PB} & \boldsymbol{PF} & \boldsymbol{C}^T & \boldsymbol{PG} & \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_p \\ * & -\boldsymbol{\Gamma}_u & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & -\boldsymbol{\Gamma}_d & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_r & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & * & -\boldsymbol{I}_r & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & * & * & -\varepsilon_1 & & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \\ * & * & * & * & * & * & & -\varepsilon_p \end{array}\right] < 0 \tag{54}$$

where

$$\boldsymbol{\Phi} = \boldsymbol{PA} + \boldsymbol{A}^T\boldsymbol{P} \tag{55}$$

the matrices

$$\boldsymbol{P} = \mathrm{diag}\left[\begin{array}{cccc} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \cdots & \boldsymbol{P}_p \end{array}\right] \tag{56}$$

$$\boldsymbol{\Gamma}_u = \mathrm{diag}\left[\begin{array}{cccc} \gamma_1 & \gamma_2 & \cdots & \gamma_p \end{array}\right] \tag{57}$$

$$\boldsymbol{\Gamma}_d = \mathrm{diag}\left[\begin{array}{cccc} \lambda_1 & \lambda_2 & \cdots & \lambda_p \end{array}\right] \tag{58}$$

are structured matrix variables, and all system matrix parameter structures are given in (13)–(17).

*Proof:* Defining Lyapunov function as follows

$$v(\boldsymbol{q}(t)) = \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{q}(t)+ \\ +\int_0^t \left( \boldsymbol{y}^T(v)\boldsymbol{y}(v) - \sum_{h=1}^{p} \left(\gamma_h u_h^2 + \lambda_h \boldsymbol{d}_h^2\right)\right) \mathrm{d}v \tag{59}$$

where $v(\boldsymbol{q}(t)) > 0$, $\boldsymbol{P} = \boldsymbol{P}^T > 0$ is given in (56), and $\gamma_h > 0$, $\lambda_h > 0$, $h = 1, 2, \dots p$, are introduced in (57).

Evaluating the derivative of $v(\boldsymbol{q}(t))$ with respect to $t$ along the autonomous system trajectories, then with the notation (11), (12) it yields

$$\dot{v}(\boldsymbol{q}(t)) = \dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^{\boldsymbol{T}}(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t)+ \\ +\boldsymbol{y}^T(t)\boldsymbol{y}(t) - \left[\begin{array}{cc} \boldsymbol{u}^T(t) & \boldsymbol{d}^T(t) \end{array}\right] \boldsymbol{\Gamma} \left[\begin{array}{c} \boldsymbol{u}(t) \\ \boldsymbol{d}(t) \end{array}\right] < 0 \tag{60}$$

where, with (57), (58)

$$\boldsymbol{\Gamma} = \mathrm{diag}\left[\begin{array}{cc} \boldsymbol{\Gamma}_u & \boldsymbol{\Gamma}_d \end{array}\right] \tag{61}$$

Therefore, the substitution of whole system model equations (8), (9) into (60) gives

$$\dot{v}(\boldsymbol{q}(t)) = \boldsymbol{q}^T(t)\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{q}(t)+ \\ +\left(\boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{G}\boldsymbol{h}(\boldsymbol{q}(t)) + \boldsymbol{F}\boldsymbol{d}(t)\right)^T \boldsymbol{P}\boldsymbol{q}(t)+ \\ +\boldsymbol{q}^T(t)\boldsymbol{P}\left(\boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{G}\boldsymbol{h}(\boldsymbol{q}(t)) + \boldsymbol{F}\boldsymbol{d}(t)\right) - \\ -\left[\begin{array}{cc} \boldsymbol{u}^T(t) & \boldsymbol{d}^T(t) \end{array}\right] \left[\begin{array}{cc} \boldsymbol{\Gamma}_u & \\ & \boldsymbol{\Gamma}_d \end{array}\right] \left[\begin{array}{c} \boldsymbol{u}(t) \\ \boldsymbol{d}(t) \end{array}\right] < 0 \tag{62}$$

Subsequently, using the inequality (20) with $\boldsymbol{X} = \boldsymbol{I}$, it can be written

$$\boldsymbol{h}^T(\boldsymbol{q}(t))\boldsymbol{G}^T\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{G}\boldsymbol{h}(\boldsymbol{q}(t)) \le \\ \le \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{h}^T(\boldsymbol{q}(t))\boldsymbol{h}(\boldsymbol{q}(t)) \tag{63}$$

and exploiting the inequality (5) then (63) gives

$$\boldsymbol{h}^T(\boldsymbol{q}(t))\boldsymbol{G}^T\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{G}\boldsymbol{h}(\boldsymbol{q}(t)) \le \\ \le \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\sum_{l=1}^{p}\varepsilon_l^{-1}\boldsymbol{w}_l^T\boldsymbol{w}_h\boldsymbol{q}(t) \tag{64}$$

It is simple to see that introducing the notation

$$\boldsymbol{q}_c^{\bullet T}(t) = \left[\begin{array}{ccc} \boldsymbol{q}^T(t) & \boldsymbol{u}^T(t) & \boldsymbol{d}^T(t) \end{array}\right] \tag{65}$$

negative (62) imply that

$$\dot{v}(\boldsymbol{q}(t)) \le \boldsymbol{q}_c^{\bullet T}(t)\boldsymbol{P}_c^{\bullet}\boldsymbol{q}_c^{\bullet}(t) < 0 \tag{66}$$

where

$$\boldsymbol{P}_c^{\bullet} = \boldsymbol{P}_{c1}^{\bullet} + \boldsymbol{P}_{c2}^{\bullet} + \boldsymbol{P}_{c3}^{\bullet} < 0 \tag{67}$$

$$\boldsymbol{P}_{c1}^{\bullet} = \left[\begin{array}{ccc} \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} & \boldsymbol{P}\boldsymbol{B} & \boldsymbol{P}\boldsymbol{F} \\ * & -\boldsymbol{\Gamma}_u & \boldsymbol{0} \\ * & * & -\boldsymbol{\Gamma}_d \end{array}\right] \tag{68}$$

$$\boldsymbol{P}_{c2}^{\bullet} = \left[\begin{array}{ccc} \boldsymbol{C}^T\boldsymbol{C} + \boldsymbol{P}\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} \\ * & \boldsymbol{0} & \boldsymbol{0} \\ * & * & \boldsymbol{0} \end{array}\right] \tag{69}$$

$$\boldsymbol{P}_{c3}^{\bullet} = \sum_{l=1}^{p}\boldsymbol{P}_{c3l}^{\bullet} = \sum_{l=1}^{p}\left[\begin{array}{ccc} \boldsymbol{w}_l^T\varepsilon_l^{-1}\boldsymbol{w}_l & \boldsymbol{0} & \boldsymbol{0} \\ * & \boldsymbol{0} & \boldsymbol{0} \\ * & * & \boldsymbol{0} \end{array}\right] \tag{70}$$

Since it yields

$$\boldsymbol{P}_{c1}^{\bullet} = \left[\begin{array}{c} \boldsymbol{C}^T \\ \boldsymbol{0} \\ \boldsymbol{0} \end{array}\right]\left[\begin{array}{ccc} \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} \end{array}\right] \ge 0 \tag{71}$$

$$\boldsymbol{P}_{c2}^{\bullet} = \left[\begin{array}{c} \boldsymbol{P}\boldsymbol{G} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{array}\right]\left[\begin{array}{ccc} \boldsymbol{G}^T\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} \end{array}\right] \ge 0 \tag{72}$$

$$\boldsymbol{P}_{c3l}^{\bullet} = \left[\begin{array}{c} \boldsymbol{w}_l \\ \boldsymbol{0} \\ \boldsymbol{0} \end{array}\right]\varepsilon_l^{-1}\left[\begin{array}{ccc} \boldsymbol{w}_l^T & \boldsymbol{0} & \boldsymbol{0} \end{array}\right] \ge 0 \tag{73}$$

applying Schur complement property to (71)–(73) then (67) implies (54). This concludes the proof. ■

Inserting the closed-loop system matrix $\boldsymbol{A}_c \in I\!R^{n\times n}$ instead the system matrix $\boldsymbol{A}$ in (54), where

$$\boldsymbol{A}_c = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} \tag{74}$$

the bilinear matrix inequality is obtained. To transform this BMI into LMI, the next new theorem is proposed.

*Theorem 1:* The system (8), with output given by the relation (9), is stabilized with quadratic performance via the controller (18) if there exist symmetric positive definite matrices $\boldsymbol{X}_i \in I\!R^{n_i\times n_i}$, the matrices $\boldsymbol{Y}_i \in I\!R^{m_i\times n_i}$ and positive scalars $\gamma_i, \lambda_i, \varepsilon_i \in I\!R$ such that for all $i = 1, 2, \dots, p$

$$\boldsymbol{X}_i = \boldsymbol{X}_i > 0,\ \gamma_i > 0,\ \lambda_i > 0,\ \varepsilon_i > 0 \tag{75}$$

$$\left[\begin{array}{cccccccc} \widetilde{\boldsymbol{\Phi}} & \boldsymbol{B} & \boldsymbol{F} & \boldsymbol{X}\boldsymbol{C}^T & \boldsymbol{G} & \boldsymbol{X}\boldsymbol{w}_1 & \cdots & \boldsymbol{X}\boldsymbol{w}_p \\ * & -\boldsymbol{\Gamma}_u & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & -\boldsymbol{\Gamma}_d & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_r & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & * & -\boldsymbol{I}_r & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & * & * & -\varepsilon_1 & & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \\ * & * & * & * & * & * & & -\varepsilon_p \end{array}\right] < 0 \tag{76}$$

where

$$\widetilde{\boldsymbol{\Phi}} = \boldsymbol{X}\boldsymbol{A}^T + \boldsymbol{A}\boldsymbol{X} - \boldsymbol{Y}^T\boldsymbol{B}^T - \boldsymbol{B}\boldsymbol{Y} \tag{77}$$

the matrices

$$\boldsymbol{X} = \mathrm{diag}\left[\begin{array}{cccc} \boldsymbol{X}_1 & \boldsymbol{X}_2 & \cdots & \boldsymbol{X}_p \end{array}\right] \tag{78}$$

$$\boldsymbol{Y} = \mathrm{diag}\left[\begin{array}{cccc} \boldsymbol{Y}_1 & \boldsymbol{Y}_2 & \cdots & \boldsymbol{Y}_p \end{array}\right] \tag{79}$$

and the matrices $\boldsymbol{\Gamma}_u$, $\boldsymbol{\Gamma}_d$ given in (57), (58), respectively, are structured matrix variables, and the system matrix parameter structures are specified in (13)–(17).

If the above conditions hold, the set of control gain matrices is given by

$$\boldsymbol{K} = \boldsymbol{Y}\boldsymbol{X}^{-1} = \left[\begin{array}{cccc} \boldsymbol{k}_1^T & \boldsymbol{k}_2^T & \cdots & \boldsymbol{k}_p^T \end{array}\right] \tag{80}$$

*Proof:* Inserting the closed-loop system matrix (74) into (55) gives

$$\boldsymbol{\Phi} = \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{K} + \boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{K}^T\boldsymbol{B}^T\boldsymbol{P} \tag{81}$$

Then, defining the transform matrix

$$\boldsymbol{T} = \operatorname{diag}\begin{bmatrix} \boldsymbol{P}^{-1} & \boldsymbol{I}_r & \boldsymbol{I}_r & \boldsymbol{I}_r & \boldsymbol{I}_r & 1 & \cdots & 1 \end{bmatrix} \tag{82}$$

and pre-multiplying the left hand as well as the right hand side of (54) by (82), the next LMI is obtained

$$\begin{bmatrix} \widetilde{\boldsymbol{\Phi}} & \boldsymbol{B} & \boldsymbol{F} & \boldsymbol{P}^{-1}\boldsymbol{C}^T & \boldsymbol{G} & \boldsymbol{P}^{-1}\boldsymbol{w}_1 & \cdots & \boldsymbol{P}^{-1}\boldsymbol{w}_p \\ * & -\boldsymbol{\Gamma}_u & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & -\boldsymbol{\Gamma}_d & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_r & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & * & -\boldsymbol{I}_r & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & * & * & -\varepsilon_1 & & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \\ * & * & * & * & * & * & & -\varepsilon_p \end{bmatrix} < 0 \tag{83}$$

where

$$\widetilde{\boldsymbol{\Phi}} = \boldsymbol{P}^{-1}\boldsymbol{A}^T - \boldsymbol{P}^{-1}\boldsymbol{K}^T\boldsymbol{B}^T + \boldsymbol{A}\boldsymbol{P}^{-1} - \boldsymbol{B}\boldsymbol{K}\boldsymbol{P}^{-1} \tag{84}$$

Introducing the LMI variables

$$\boldsymbol{P}^{-1} = \boldsymbol{X}, \quad \boldsymbol{K}\boldsymbol{P}^{-1} = \boldsymbol{Y} \tag{85}$$

then (85) implies (79), and (83), (84) implies (76), (77), respectively. This concludes the proof. ■

Note, now the optimization problem in Theorem 1 can be solved using the standard LMI solvers.

## VI. Enhanced Design Conditions

The idea of separating the Lyapunov matrix $\boldsymbol{P}$ from the system matric parameters is based on the method of Krasovskii and the theory of slack matrices [13]. In short, for the linear large-scale systems this method lies the next new stability conditions, formulated with respect to the subsystem interactions quadratic performances.

*Theorem 2:* The autonomous system from (8) is asymptotically stable with bounded quadratic performance if for given positive $\delta \in I\!R$ there exist symmetric positive definite matrices $\boldsymbol{P}_i, \boldsymbol{V}_i \in I\!R^{n_i \times n_i}$ and positive scalars $\gamma_i, \lambda_i, \epsilon_i \in I\!R$ such that for $i = 1, 2, \ldots, p$

$$\boldsymbol{P}_i = \boldsymbol{P}_i > 0,\ \boldsymbol{V}_i = \boldsymbol{V}_i > 0,\ \gamma_i > 0,\ \lambda_i > 0,\ \epsilon_i > 0 \tag{86}$$

$$\begin{bmatrix} \boldsymbol{\Lambda} & \boldsymbol{V}\boldsymbol{B} & \boldsymbol{V}\boldsymbol{F} & \boldsymbol{\Psi} & \boldsymbol{C}^T & \boldsymbol{V}\boldsymbol{G} & \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_p \\ * & -\boldsymbol{\Gamma}_u & \boldsymbol{0} & \delta\boldsymbol{B}^T\boldsymbol{V} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & -\boldsymbol{\Gamma}_d & \delta\boldsymbol{F}^T\boldsymbol{V} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & -\boldsymbol{\Pi} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & * & -\boldsymbol{I}_r & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & * & * & -\boldsymbol{I}_r & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & * & * & * & -\epsilon_1 & & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \\ * & * & * & * & * & * & * & & -\epsilon_p \end{bmatrix} < 0 \tag{87}$$

where

$$\boldsymbol{\Lambda} = \boldsymbol{V}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{V} \tag{88}$$

$$\boldsymbol{\Psi} = \boldsymbol{P} - \boldsymbol{V} + \delta\boldsymbol{A}^T\boldsymbol{V} \tag{89}$$

$$\boldsymbol{\Pi} = 2\delta\boldsymbol{V} - \delta^2\boldsymbol{V}\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{V} \tag{90}$$

the matrix

$$\boldsymbol{V} = \operatorname{diag}\begin{bmatrix} \boldsymbol{V}_1 & \boldsymbol{V}_2 & \cdots & \boldsymbol{V}_p \end{bmatrix} \tag{91}$$

and the matrices $\boldsymbol{P}$, $\boldsymbol{\Gamma}_u$, $\boldsymbol{\Gamma}_d$ given in (56), (57), (58), respectively, are structured matrix variables, and the system matrix parameter structures are specified in (13)–(17).

*Proof:* Since (8) implies

$$\boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{G}\boldsymbol{h}(\boldsymbol{q}(t)) + \boldsymbol{F}\boldsymbol{d}(t) - \dot{\boldsymbol{q}}(t) = \boldsymbol{0} \tag{92}$$

then with positive definite symmetric block diagonal matrices $\boldsymbol{V}_1^\diamond$, $\boldsymbol{V}_2^\diamond \in I\!R^{n \times n}$ it yields

$$\left(\boldsymbol{q}^T(t)\boldsymbol{V}_1^\diamond + \dot{\boldsymbol{q}}^T(t)\boldsymbol{V}_2^\diamond\right)\begin{pmatrix} \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \\ +\boldsymbol{G}\boldsymbol{h}(\boldsymbol{q}(t)) + \boldsymbol{F}\boldsymbol{d}(t) - \dot{\boldsymbol{q}}(t) \end{pmatrix} = 0 \tag{93}$$

Then, adding (93) and the transpose of (93) to (60) and subsequently substituting (9) in (60) results in

$$\begin{aligned} \dot{v}(\boldsymbol{q}(t)) =\ & \dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t) + \boldsymbol{q}^T(t)\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{q}(t) + \\ & + \begin{pmatrix} \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \\ +\boldsymbol{G}\boldsymbol{h}(\boldsymbol{q}(t)) + \boldsymbol{F}\boldsymbol{d}(t) - \dot{\boldsymbol{q}}(t) \end{pmatrix}^T (\boldsymbol{V}_1^\diamond\boldsymbol{q}(t) + \boldsymbol{V}_2^\diamond\dot{\boldsymbol{q}}(t)) + \\ & + \left(\boldsymbol{q}^T(t)\boldsymbol{V}_1^\diamond + \dot{\boldsymbol{q}}^T(t)\boldsymbol{V}_2^\diamond\right)\begin{pmatrix} \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \\ +\boldsymbol{G}\boldsymbol{h}(\boldsymbol{q}(t)) + \boldsymbol{F}\boldsymbol{d}(t) - \dot{\boldsymbol{q}}(t) \end{pmatrix} - \\ & - \begin{bmatrix} \boldsymbol{u}^T(t) & \boldsymbol{d}^T(t) \end{bmatrix}\begin{bmatrix} \boldsymbol{\Gamma}_u & \\ & \boldsymbol{\Gamma}_d \end{bmatrix}\begin{bmatrix} \boldsymbol{u}(t) \\ \boldsymbol{d}(t) \end{bmatrix} < 0 \end{aligned} \tag{94}$$

Subsequently, using the inequality (20) with $\boldsymbol{X} = \boldsymbol{I}$, it can be written

$$\begin{aligned} & \boldsymbol{h}^T(\boldsymbol{q}(t))\boldsymbol{G}^T\boldsymbol{V}_1^\diamond\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{V}_1^\diamond\boldsymbol{G}\boldsymbol{h}(\boldsymbol{q}(t)) \le \\ & \le \boldsymbol{h}^T(\boldsymbol{q}(t))\boldsymbol{h}(\boldsymbol{q}(t)) + \boldsymbol{q}^T(t)\boldsymbol{V}_1^\diamond\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{V}_1^\diamond\boldsymbol{q}(t) \end{aligned} \tag{95}$$

$$\begin{aligned} & \boldsymbol{h}^T(\boldsymbol{q}(t))\boldsymbol{G}^T\boldsymbol{V}_2^\diamond\dot{\boldsymbol{q}}(t) + \dot{\boldsymbol{q}}^T(t)\boldsymbol{V}_2^\diamond\boldsymbol{G}\boldsymbol{h}(\boldsymbol{q}(t)) \le \\ & \le \boldsymbol{h}^T(\boldsymbol{q}(t))\boldsymbol{h}(\boldsymbol{q}(t)) + \dot{\boldsymbol{q}}^T(t)\boldsymbol{V}_2^\diamond\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{V}_2^\diamond\dot{\boldsymbol{q}}(t) \end{aligned} \tag{96}$$

and, exploiting (5), then (95), (96) give

$$\begin{aligned} & \boldsymbol{h}^T(\boldsymbol{q}(t))\boldsymbol{G}^T\boldsymbol{V}_1^\diamond\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{V}_1^\diamond\boldsymbol{G}\boldsymbol{h}(\boldsymbol{q}(t)) \le \\ & \le \boldsymbol{q}^T(t)\sum_{l=1}^{p}\varepsilon_l^{-1}\boldsymbol{w}_l^T\boldsymbol{w}_l\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{V}_1^\diamond\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{V}_1^\diamond\boldsymbol{q}(t) \end{aligned} \tag{97}$$

$$\begin{aligned} & \boldsymbol{h}^T(\boldsymbol{q}(t))\boldsymbol{G}^T\boldsymbol{V}_2^\diamond\dot{\boldsymbol{q}}(t) + \dot{\boldsymbol{q}}^T(t)\boldsymbol{V}_2^\diamond\boldsymbol{G}\boldsymbol{h}(\boldsymbol{q}(t)) \le \\ & \le \boldsymbol{q}^T(t)\sum_{l=1}^{p}\varepsilon_l^{-1}\boldsymbol{w}_l^T\boldsymbol{w}_l\boldsymbol{q}(t) + \dot{\boldsymbol{q}}^T(t)\boldsymbol{V}_2^\diamond\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{V}_2^\diamond\dot{\boldsymbol{q}}(t) \end{aligned} \tag{98}$$

respectively. Thus, introducing the notation

$$\boldsymbol{q}_c^{\diamond T}(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \boldsymbol{u}^T(t) & \boldsymbol{d}^T(t) & \dot{\boldsymbol{q}}^T(t) \end{bmatrix} \tag{99}$$

(94) can be rewritten as

$$\dot{v}(\boldsymbol{q}(t)) \le \boldsymbol{q}_c^{\diamond T}(t)\boldsymbol{P}_c^\diamond\boldsymbol{q}_c^\diamond(t) < 0 \tag{100}$$

where

$$\boldsymbol{P}_c^\diamond = \boldsymbol{P}_{c1}^\diamond + \boldsymbol{P}_{c2}^\diamond + \boldsymbol{P}_{c3}^\diamond < 0 \tag{101}$$

$$\boldsymbol{P}_{c1}^\diamond = \\ = \begin{bmatrix} \boldsymbol{V}_1^\diamond\boldsymbol{A}+\boldsymbol{A}^T\boldsymbol{V}_1^\diamond & \boldsymbol{V}_1^\diamond\boldsymbol{B} & \boldsymbol{V}_1^\diamond\boldsymbol{F} & \boldsymbol{P}-\boldsymbol{V}_1^\diamond+\boldsymbol{A}^T\boldsymbol{V}_2^\diamond \\ * & -\boldsymbol{\Gamma}_u & \boldsymbol{0} & \boldsymbol{B}^T\boldsymbol{V}_2^\diamond \\ * & * & -\boldsymbol{\Gamma}_d & \boldsymbol{F}^T\boldsymbol{V}_2^\diamond \\ * & * & * & -2\boldsymbol{V}_2^\diamond+\boldsymbol{V}_2^\diamond\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{V}_2^\diamond \end{bmatrix} \tag{102}$$

$$\boldsymbol{P}_{c2}^\diamond = \begin{bmatrix} \boldsymbol{C}^T\boldsymbol{C}+\boldsymbol{V}_1^\diamond\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{V}_1^\diamond & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & \boldsymbol{0} \end{bmatrix} \tag{103}$$

$$\boldsymbol{P}_{c3}^\diamond = \sum_{l=1}^{p}\boldsymbol{P}_{c3l}^\diamond = \sum_{l=1}^{p}\begin{bmatrix} \boldsymbol{w}_l^T\epsilon_l^{-1}\boldsymbol{w}_l & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & \boldsymbol{0} \end{bmatrix} \tag{104}$$

$$\epsilon_l^{-1} = 2\varepsilon_l^{-1} \tag{105}$$

Analogously, setting

$$\boldsymbol{V}_1^\diamond = \boldsymbol{V}, \qquad \boldsymbol{V}_2^\diamond = \delta\boldsymbol{V} \tag{106}$$

and applying the Schur complement property to (103), (104), then (101) implies (87). This concludes the proof. ■

The following theorem presents the design of a continuous state feedback controller to decentralized stabilization of the system (9).

*Theorem 3:* The system (8), with output given by the relation (9), is stabilized with quadratic performance via the controller (18) if there exist symmetric positive definite matrices $\boldsymbol{T}_i, \boldsymbol{Z}_i \in I\!R^{n_i\times n_i}$, the matrices $\boldsymbol{W}_i \in I\!R^{m_i\times n_i}$ and positive scalars $\gamma_i, \lambda_i, \epsilon_i \in I\!R$ such that for all $i = 1, 2, \ldots, p$

$$\boldsymbol{T}_i = \boldsymbol{T}_i > 0,\ \boldsymbol{Z}_i = \boldsymbol{Z}_i > 0,\ \gamma_i > 0,\ \lambda_i > 0,\ \epsilon_i > 0 \tag{107}$$

$$\begin{bmatrix} \widetilde{\boldsymbol{\Lambda}} & \boldsymbol{B} & \boldsymbol{F} & \widetilde{\boldsymbol{\Psi}} & \boldsymbol{Z}\boldsymbol{C}^T & \boldsymbol{G} & \boldsymbol{Z}\boldsymbol{w}_1 & \cdots & \boldsymbol{Z}\boldsymbol{w}_p \\ * & -\boldsymbol{\Gamma}_u & \boldsymbol{0} & \delta\boldsymbol{B}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & -\boldsymbol{\Gamma}_d & \delta\boldsymbol{F}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & -\widetilde{\boldsymbol{\Pi}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & * & -\boldsymbol{I}_r & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & * & * & -\boldsymbol{I}_r & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & * & * & * & -\epsilon_1 & & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \\ * & * & * & * & * & * & * & & -\epsilon_p \end{bmatrix} < 0 \tag{108}$$

where

$$\widetilde{\boldsymbol{\Lambda}} = \boldsymbol{A}\boldsymbol{Z} - \boldsymbol{B}\boldsymbol{W} + \boldsymbol{Z}\boldsymbol{A}^T - \boldsymbol{W}^T\boldsymbol{B}^T \tag{109}$$

$$\widetilde{\boldsymbol{\Psi}} = \boldsymbol{T} - \boldsymbol{Z} + \delta(\boldsymbol{Z}\boldsymbol{A}^T - \boldsymbol{W}^T\boldsymbol{B}^T) \tag{110}$$

$$\widetilde{\boldsymbol{\Pi}} = 2\delta\boldsymbol{Z} - \delta^2\boldsymbol{G}\boldsymbol{G}^T \tag{111}$$

the matrices

$$\boldsymbol{T} = \operatorname{diag}\left[\ \boldsymbol{T}_1 \quad \boldsymbol{T}_2 \quad \cdots \quad \boldsymbol{T}_p\ \right] \tag{112}$$

$$\boldsymbol{Z} = \operatorname{diag}\left[\ \boldsymbol{Z}_1 \quad \boldsymbol{Z}_2 \quad \cdots \quad \boldsymbol{Z}_p\ \right] \tag{113}$$

$$\boldsymbol{W} = \operatorname{diag}\left[\ \boldsymbol{W}_1 \quad \boldsymbol{W}_2 \quad \cdots \quad \boldsymbol{W}_p\ \right] \tag{114}$$

and the matrices $\boldsymbol{\Gamma}_u$, $\boldsymbol{\Gamma}_d$ given in (57), (58), respectively, are structured matrix variables, and the system matrix parameter structures are specified in (13)–(17).

If the above conditions hold, the set of control gain matrices is given by

$$\boldsymbol{K} = \boldsymbol{W}\boldsymbol{Z}^{-1} = \left[\ \boldsymbol{k}_1^T \quad \boldsymbol{k}_2^T \quad \cdots \quad \boldsymbol{k}_p^T\ \right] \tag{115}$$

*Proof:* Inserting the closed-loop system matrix (74) into (88), (89) gives

$$\boldsymbol{\Lambda} = \boldsymbol{V}\boldsymbol{A} - \boldsymbol{V}\boldsymbol{B}\boldsymbol{K} + \boldsymbol{A}^T\boldsymbol{V} - \boldsymbol{K}^T\boldsymbol{B}^T\boldsymbol{V} \tag{116}$$

$$\boldsymbol{\Psi} = \boldsymbol{P} - \boldsymbol{V} + \delta(\boldsymbol{A}^T\boldsymbol{V} - \boldsymbol{K}^T\boldsymbol{B}^T\boldsymbol{V}) \tag{117}$$

$$\boldsymbol{\Pi} = 2\delta\boldsymbol{V} - \delta^2\boldsymbol{V}\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{V} \tag{118}$$

Since the matrix $\boldsymbol{V}$ is supposed to be positive definite, it can be set up the next transform matrix

$$\boldsymbol{T} = \operatorname{diag}\left[\ \boldsymbol{V}^{-1} \quad \boldsymbol{I}_r \quad \boldsymbol{I}_r \quad \boldsymbol{V}^{-1} \quad \boldsymbol{I}_r \quad \boldsymbol{I}_r \quad 1 \quad \cdots \quad 1\ \right] \tag{119}$$

Pre-multiplying the left hand and the right hand side of (87) by (119), then it yields

$$\begin{bmatrix} \widetilde{\boldsymbol{\Lambda}} & \boldsymbol{B} & \boldsymbol{F} & \widetilde{\boldsymbol{\Psi}} & \boldsymbol{V}^{-1}\boldsymbol{C}^T & \boldsymbol{G} & \boldsymbol{w}_1^\diamond & \cdots & \boldsymbol{w}_p^\diamond \\ * & -\boldsymbol{\Gamma}_u & \boldsymbol{0} & \delta\boldsymbol{B}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & -\boldsymbol{\Gamma}_d & \delta\boldsymbol{F}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & -\widetilde{\boldsymbol{\Pi}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & * & -\boldsymbol{I}_r & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & * & * & -\boldsymbol{I}_r & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ * & * & * & * & * & * & -\epsilon_1 & & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \\ * & * & * & * & * & * & * & & -\epsilon_p \end{bmatrix} < 0 \tag{120}$$

where

$$\widetilde{\boldsymbol{\Lambda}} = \boldsymbol{A}\boldsymbol{V}^{-1} - \boldsymbol{B}\boldsymbol{K}\boldsymbol{V}^{-1} + \boldsymbol{V}^{-1}\boldsymbol{A}^T - \boldsymbol{V}^{-1}\boldsymbol{K}^T\boldsymbol{B}^T \tag{121}$$

$$\widetilde{\boldsymbol{\Psi}} = \boldsymbol{V}^{-1}\boldsymbol{P}\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1} + \delta(\boldsymbol{V}^{-1}\boldsymbol{A}^T - \boldsymbol{V}^{-1}\boldsymbol{K}^T\boldsymbol{B}^T) \tag{122}$$

$$\widetilde{\boldsymbol{\Pi}} = 2\delta\boldsymbol{V}^{-1} - \delta^2\boldsymbol{G}\boldsymbol{G}^T \tag{123}$$

$$\boldsymbol{w}_l^\diamond = \boldsymbol{V}^{-1}\boldsymbol{w}_l,\ l = 1, 2, \ldots p \tag{124}$$

Introducing the LMI variables

$$\boldsymbol{V}^{-1} = \boldsymbol{Z}, \quad \boldsymbol{K}\boldsymbol{V}^{-1} = \boldsymbol{W}, \quad \boldsymbol{V}^{-1}\boldsymbol{P}\boldsymbol{V}^{-1} = \boldsymbol{T} \tag{125}$$

then (120)-(123) implies (108)–(111), respectively. This concludes the proof. ■

In order to make this result applicable and operational, it is necessary to give the parameter $\delta$ and verify the obtained $H_\infty$ quadratic constraints. In addition, the size of this parameter can be used for tuning of the dynamics of the closed-loop system responses.

## VII. Illustrative Example

To demonstrate the algorithm properties, the next subsystem parameters for $i = 1, 2, 3$ are used [18]

$$\boldsymbol{A}_i = \begin{bmatrix} -12.50 & 0.00 & -5.21 & 0.00 \\ 3.33 & -3.33 & 0.00 & 0.00 \\ 0.00 & 6.00 & -0.05 & -6.00 \\ 0.00 & 0.00 & 1.10 & 0.00 \end{bmatrix}, \ \boldsymbol{b}_i = \begin{bmatrix} 12.5 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}$$

$$\boldsymbol{c}_i^T = [\,0 \quad 0 \quad 1 \quad 0\,], \quad \boldsymbol{f}_i^T = [\,0 \quad 0 \quad -6 \quad 0\,]$$

and

$$\boldsymbol{G}_{ih} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.55 & 0 \end{bmatrix}, \ \boldsymbol{g}_x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.55 \end{bmatrix}$$

$$\boldsymbol{G} = \mathrm{diag}\,[\,\boldsymbol{g}_x \quad \boldsymbol{g}_x \quad \boldsymbol{g}_x\,]$$

$$\boldsymbol{w}_1^T = [\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0\,]$$

$$\boldsymbol{w}_2^T = [\,0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0\,]$$

$$\boldsymbol{w}_3^T = [\,0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,]$$

Thus, solving (75), (76) with respect to the LMI matrix variables $\boldsymbol{X}_i, \boldsymbol{Y}_i, \gamma_i, \lambda_i, \varepsilon_i, i = 1, 2, 3$ using SeDuMi package for Matlab [21], the feedback gain matrix design problem was feasible with the results

$$\boldsymbol{X}_i = \begin{bmatrix} 14.7389 & 2.7960 & -1.9062 & 3.6230 \\ * & 5.0205 & -1.7327 & 3.4803 \\ * & * & 2.2244 & -0.6127 \\ * & * & * & 3.6596 \end{bmatrix}$$

$$\boldsymbol{Y}_i^T = [\,-12.9516 \quad 1.4183 \quad 0.1961 \quad -3.4268\,]$$

$$\gamma_i = 16.9863, \ \lambda_i = 15.5306, \ \varepsilon_i = 11.5833$$

According to these matrix parameters, the local control laws are constructed with the gain vectors

$$\boldsymbol{k}_i^T = [\,-0.4870 \quad 3.5065 \quad 1.4243 \quad -3.5505\,]$$

resulting in the stable decentralized state control, characteristic by the subsystem closed-loop matrix eigenvalues spectrum

$$\rho(\boldsymbol{A}_{ch}) = \{-0.2646 \ -3.2126 \ -3.1578 \pm 11.9004\mathrm{i}\}$$

Solving (107), (108) for $\delta = 0.2$ with respect to the LMI matrix variables $\boldsymbol{T}_i$, $\boldsymbol{Z}_i$, $\boldsymbol{Y}_i$, $\gamma_i$, $\lambda_i$, $\epsilon_i$, $i = 1, 2, 3$, the feasible solution gives out the common positive-definite matrix variables

$$\boldsymbol{T}_i = \begin{bmatrix} 48.1661 & -4.5647 & -2.5138 & 4.2776 \\ * & 5.3518 & -3.2610 & 1.7874 \\ * & * & 4.0874 & -1.1766 \\ * & * & * & 1.8135 \end{bmatrix}$$

$$\boldsymbol{Z}_i = \begin{bmatrix} 9.5165 & 0.7452 & -1.5398 & 1.1538 \\ * & 3.4365 & -2.4504 & 1.5058 \\ * & * & 3.1948 & -1.0502 \\ * & * & * & 1.4237 \end{bmatrix}$$

and the LMI parameters

$$\boldsymbol{W}_i^T = [\,-6.7570 \quad 2.1314 \quad 0.2832 \quad -0.5702\,]$$

$$\gamma_i = 22.1502, \ \lambda_i = 37.2683, \ \epsilon_i = 10.4566$$

Note that by increasing the value of the tuning parameter $\delta$ the LMI solution may become infeasible.

The local state control with the obtained gain matrix

$$\boldsymbol{k}_i^T = [\,-0.5540 \quad 1.9349 \quad 0.8510 \quad -1.3508\,]$$

insures the global system stability with decentralized closed-loop system matrix eigenvalues spectrum

$$\rho(\boldsymbol{A}_{ch}) = \{-1.0044 \ -3.4075 \ -2.2717 \pm 8.8048\mathrm{i}\}$$

That, in the both cases, the resulting system time responses have small relative damping is not given by the attribute of the presented algorithms but implies from the characteristic properties of multiarea power systems.

## VIII. Concluding Remarks

Decentralized robust control design for large scale systems with relevant subsystem interactions is formulated in the paper as an optimization problem and solved by LMIs. A conveyed characterization for the interaction bounds is presented and the sufficient condition for stabilizing decentralized robust control design are newly originated in the bounded real lemma as well as in the enhanced bounded real lemma structure, respectively. Since the theorems are newly derived, the proofs was necessary to include due to their original contributions.

The optimization principe, involving structured matrix variables in the linear matrix inequalities, takes into account the interaction bounds and the resulted decomposition gives enough flexibility to allow the inclusion of more general subsystem interaction structures and different output channels measurement gains.

The feasibility and effectiveness of enhanced bounded real lemma based control design are demonstrated using a multi-area model of the power system. It was shown that the global system can be locally asymptotically stabilizable by the decentralized state feedback laws, where application of the tuning parameter can improve dynamic system responses.

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## References

[1] D. Krokavec and A. Filasová, "Decentralized state-space control involving subsystem interactions," in *Proc. 8th Int. Conf. on Systems ICONS 2013*, Sevilla, Spain, pp. 13-18, 2013.

[2] A.C. Antoulas, *Approximation of Large-Scale Dynamical Systems*, Philadelphia: SIAM, PE, USA, 2005.

[3] L. Bakule, "Decentralized control. An overview," *Annual Reviews in Control*, vol. 32, no. 1, pp. 87-98, 2008.

[4] G.K. Bekefadu and I. Erlich, "Robust decentralized controller design for power systems using convex optimization involving LMIs," in *Prepr. 16$^{th}$ IFAC Word Congress*, Prag, Czech Republic, pp. 1743-1743, 2005.

[5] B. Boyd, L. El Ghaoui, E. Peron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Philadelphia: SIAM, PE, USA, 1994.

[6] W.C. Chan and Y.Y. Hsu, "Automatic generation control of interconnected power systems using variable-structure controllers," *IEE Proc. C*, vol. 128, no. 5, pp. 269-279, 1981.

[7] N. Chen, M. Ikeda, and W. Gu, "Design of robust $H_\infty$ control for interconnected systems: A homotopy method," *Int. J. Control, Automation, and Systems*, vol. 3, no. 2, pp. 143-151, 2005.

[8] C. Cheng, B. Tang, Y. Cao, and Y. Sun, "Decentralized robust $H_\infty$ control of uncertain large-scale systems with state-delays. LMIs approach," *Proc. American Control Conference*, Philadelphia, PE, USA, pp. 3111-3115, 1998.

[9] C. Dou, J. Yang, X. Li, T. Gui, and Y. Bi, "Decentralized coordinated control for large power system based on transient stability assessment," *Int. J. Electrical Power & Energy Systems*, vol. 46, no. 1, pp. 153-162, 2013.

[10] O.I. Elgert and C.E. Fosha, "Optimum megawatt-frequency control of multiarea electric energy system," *IEEE Trans. Power Apparatus and Systems*, vol. 89, no. 4, pp. 556-563, 1970.

[11] D.G. Feingold and R.S. Varga, "Block diagonally dominant matrices and generalizations of the Gerschgorin circle theorem," *Pacific J. Math.*, vol. 12, no. 4, 1241-1250, 1962.

[12] A, Filasová and D. Krokavec, "Pairwise control principle in large-scale systems," *Arch. Control Sciences*, vol. 21, no. 3, pp. 227-242, 2011.

[13] A, Filasová and D. Krokavec, "Partially decentralized design principle in large-scale system control," in *Recent Advances in Robust Control. Novel Approaches and Design Methods*, A. Mueller Ed., Rijeca: InTech, Croatia, pp. 361-388, 2011.

[14] C.E. Fosha and O.I. Elgert, "The megawatt-frequency control problem. A new approach via optimal control theory," *IEEE Trans. Power Apparatus and Systems*, vol. 89, no. 4, pp. 563-577, 1970.

[15] Y. Guo, D.J. Hill and Y. Wang, "Nonlinear decentralized control of large-scale power systems," *Automatica*, vol. 36, no. 9, pp. 1275-1289, 2000.

[16] W.M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control. A Lyapunov-Based Approach*, Princeton: Princeton University Press, NJ, USA, 2008.

[17] M. Jamshidi *Large-Scale Systems: Modeling, Control and Fuzzy Logic*, Upper Saddle River: Prentice Hall, NJ, USA, 1997.

[18] D. Krokavec and A. Filasová, "Load frequency control involving subsystem interaction," in *Proc. 9th Int. Conf. Control of Power Systems CPS 2010*, Tatranske Matliare, Slovakia, pp. 1-8, 2010.

[19] J. Lunze, *Feedback Control of Large-Scale Systems*, Englewood Cliffs: Prentice Hall, NJ, USA, 1992.

[20] J. Mohammadpour and K.M. Grigoriadis, *Efficient Modeling and Control of Large-Scale Systems*, New York: Springer, NY, USA, 2010.

[21] D. Peaucelle, D. Henrion, Y. Labit, and K. Taitz, *User's Guide for SeDuMi Interface 1.04*, Toulouse: LAAS-CNRS, France, 2002.

[22] G. Pipeleers, B. Demeulenaerea, J. Sweversa, and L. Vandenbergheb, "Extended LMI characterizations for stability and performance of linear systems," *Systems & Control Letters*, vol.58, no. 7, pp. 510-518, 2009.

[23] D.D. Siljak, D.M. Stipanovic, and A.I. Zecevic, "Robust decentralized turbine/governor control using linear matrix inequalities," *IEEE Trans. Power Systems*, vol. 19, no. 3, pp, 1096-1103, 2004.

[24] Y. Wang, R. Zhou, and C. Wen, "Robust load-frequency controller design for power systems," *IEE Proc. C*, vol. 140, no. 1, pp. 11-16, 1993.

[25] G. Zhai, M. Ikeda and Y. Fujisaki, "Decentralized $H_\infty$ controller design. A matrix inequality approach using a homotopy method," *Automatica*, vol. 37, no. 4, pp. 565-572, 2001.

## APPENDIX

Considering a multi-area power system, the next analysis is based on the assumption that the electrical interconnections within each area of multi-area power system are so strong, at least in relation to ties with the neighboring areas that the whole area can be characterized only by a single frequency (see, e.g., [18] and the references therein). Therefore, it is supposed that the power equilibrium applied to the area $i$ can be written as

$$\begin{aligned} T_{Pi}\tfrac{\mathrm{d}\Delta f_i(t)}{\mathrm{d}t} + \Delta f_i(t) + K_{Pk}\Delta P_{Tk}(t) = \\ = K_{Pi}\Delta P_{Gi}(t) - K_{Pi}\Delta P_{Di}(t) \end{aligned} \tag{A.1}$$

where $T_{Pi}$ is the area model time constant (s), $\Delta f_i(t)$ is the area incremental frequency deviation (Hz), $K_{Pi}$ is the area gain (Hz/pu MW), $\Delta P_{Ti}(t)$ is the incremental change of the total real power exported from the area (Hz/pu MW), $\Delta P_{Gi}(t)$ is the incremental change in generator output (Hz/pu MW), and $\Delta P_{Di}(t)$ is the unknown load disturbance (Hz/pu MW).

If the line losses are neglected, the individual line powers can be written in the form

$$\begin{aligned} P_{Ti}(t) = \tfrac{|V_i||V_\upsilon|}{X_{\upsilon i}P_{\upsilon i}}\sin(\delta_i(t)-\delta_\upsilon(t)) = \\ = P_{Ti\upsilon\max}\sin(\delta_i(t)-\delta_\upsilon(t)) \end{aligned} \tag{A.2}$$

$$V_i(t) = |V_i|\exp(j\delta_i(t)),\ V_\upsilon(t) = |V_\upsilon|\exp(j\delta_\upsilon(t)) \tag{A.3}$$

is the terminal bus voltage of the line, and $X_{\nu i}$ is its reactance.

When the phase angles deviate from their nominal values by the amounts $\Delta\delta_i$, $\Delta\delta_\nu$, respectively, the next approximation can be obtained [14]

$$\begin{aligned} \Delta P_{Ti}(t) = \\ = \tfrac{V_i||V_\upsilon|}{X_{\upsilon i}P_{\upsilon i}}\cos(\delta_{in}(t)-\delta_{\upsilon n}(t))(\Delta\delta_i(t)-\Delta\delta_\upsilon(t)) \end{aligned} \tag{A.4}$$

$$\begin{aligned} \Delta P_{Ti}(t) = \\ = 2\pi\tfrac{|V_i||V_\upsilon|}{X_{\upsilon i}P_{\upsilon i}}\cos(\delta_{in}(t)-\delta_{\upsilon n}(t))\left\{\begin{matrix}\int_0^t \Delta f_i(r)\mathrm{d}r - \\ -\int_0^t \Delta f_\upsilon(r)\mathrm{d}r\end{matrix}\right\} \end{aligned} \tag{A.5}$$

respectively. Related to the area frequency changes, the time derivative of the individual line powers is

$$\frac{\mathrm{d}\Delta P_{Ti\upsilon}(t)}{\mathrm{d}t} = S_{i\upsilon}(\Delta f_i(t) - \Delta f_\upsilon(t)) \tag{A.6}$$

$$\frac{\mathrm{d}\Delta P_{Ti}(t)}{\mathrm{d}t} = \sum_{i\neq l} S_{il}(\Delta f_i(t) - \Delta f_l(t)) \tag{A.7}$$

respectively, where $S_{il}$ is the synchronizing coefficient (electrical stiffness of the tie line).

The incremental generated power of the area $i$ for small system variable changes around the nominal settings can be represented by the equations

$$T_{Ti}\frac{\mathrm{d}\Delta P_{Gi}(t)}{\mathrm{d}t} + \Delta P_{Gi}(t) = \Delta x_{Hi}(t) \tag{A.8}$$

$$T_{Hi}\frac{\mathrm{d}\Delta x_{Hi}(t)}{\mathrm{d}t} + \Delta x_{Hi}(t) = \Delta P_{Ci}(t) - \frac{1}{R_i}\Delta f_i(t) \tag{A.9}$$

where $T_{Ti}$ is the turbine time constant (s), $T_{Hi}$ is the governor time constant (s) (generator response is instantaneous), $R_i$ is a measure of static speed droop (Hz/pu MW), $\Delta P_{Ci}(t)$ is the incremental change of the command signal to the speed changer (control input), and $\Delta x_{Hi}(t)$ is the incremental change in the governor value position (pu MW), all with respect to the area $i$.

From the analysis made above the given formulation shows that the functioning of the multiarea power system is roughly a process with relevant interactions. The compact notation of (A.1), (A.7), (A.8), and (A.9) in the state-space form so leads to the equations [18]

$$\dot{\boldsymbol{q}}_i(t) = \boldsymbol{A}_i\boldsymbol{q}_i(t) + \boldsymbol{b}_i u_i(t) + \sum_{l=1}^{p} \boldsymbol{G}_{li}\boldsymbol{q}_i(t) + \boldsymbol{f}_i d_i(t) \qquad \text{(A.10)}$$

$$y_i(t) = \boldsymbol{c}_i^T \boldsymbol{q}_i(t) \qquad \text{(A.11)}$$

where

$$\boldsymbol{q}_i(t) = [\,\Delta x_{Hi}(t)\ \ \Delta P_{Gi}(t)\ \ \Delta f_i(t)\ \ \Delta P_{Ti}(t)\,]^T \qquad \text{(A.12)}$$

$$u_i(t) = \Delta P_{Ci}(t) \qquad \text{(A.13)}$$

$$d_i(t) = \Delta P_{Di}(t) \qquad \text{(A.14)}$$

$$\boldsymbol{A}_i = \begin{bmatrix} -\frac{1}{T_{Hi}} & 0 & -\frac{1}{R_i T_{Hi}} & 0 \\ \frac{1}{T_{Ti}} & -\frac{1}{T_{Ti}} & 0 & 0 \\ 0 & \frac{K_{Pi}}{T_{Pi}} & -\frac{1}{T_{Pi}} & -\frac{K_{Pi}}{T_{Pi}} \\ 0 & 0 & \sum_{l \neq i} S_{il} & 0 \end{bmatrix} \qquad \text{(A.15)}$$

$$\boldsymbol{G}_{li} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -S_{li} & 0 \end{bmatrix}, \quad \boldsymbol{b}_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{T_{Hi}} \end{bmatrix} \qquad \text{(A.16)}$$

$$\boldsymbol{f}_i = \begin{bmatrix} 0 \\ 0 \\ -\frac{K_{Pi}}{T_{Pi}} \\ 0 \end{bmatrix}, \quad \boldsymbol{c}_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \text{(A.17)}$$

More details, or multiarea model structure modifications, can be found, e.g., in [6], [10].

Under above given model parameters, the stability of the overall system can be studied by the stability properties of all subsystems, and by global features of all subsystems interactions.